The British Journal for the Philosophy of Science Advance Access published September 26, 2016

Brit. J. Phil. Sci. **0** (2016), 1–44

Does Homotopy Type Theory Provide a Foundation for Mathematics?

James Ladyman and Stuart Presnell

ABSTRACT

Homotopy Type Theory (HoTT) is a putative new foundation for mathematics grounded in constructive intensional type theory that offers an alternative to the foundations provided by ZFC set theory and category theory. This article explains and motivates an account of how to define, justify, and think about HoTT in a way that is self-contained, and argues that, so construed, it is a candidate for being an autonomous foundation for mathematics. We first consider various questions that a foundation for mathematics might be expected to answer, and find that many of them are not answered by the standard formulation of HoTT as presented in the 'HoTT Book'. More importantly, the presentation of HoTT given in the HoTT Book is not autonomous since it explicitly depends upon other fields of mathematics, in particular homotopy theory. We give an alternative presentation of HoTT that does not depend upon ideas from other parts of mathematics, and in particular makes no reference to homotopy theory (but is compatible with the homotopy interpretation), and argue that it is a candidate autonomous foundation for mathematics. Our elaboration of HoTT is based on a new interpretation of types as mathematical concepts, which accords with the intensional nature of the type theory.

doi:10.1093/bjps/axw006

1 Introduction

Homotopy Type Theory (HoTT) is first and foremost a research programme within mathematics that connects algebraic topology with logic, computer science, and category theory. Its name derives from the way it integrates homotopy theory (which concerns spaces, points, and paths) and formal type theory (as pioneered by Russell, Church, and Gödel, and developed in computer science) by interpreting types as spaces and terms of them as points in those spaces. Hence, the extant text on the theory (The Univalent Foundations Program [2013]; hereafter the 'HoTT Book'), hereafter the 'HoTT Book', involves homotopy theory throughout.

The authors of the HoTT Book are concerned to develop and promote HoTT for working mathematicians, and to establish it as a foundation for mathematics—the subtitle of the HoTT Book is 'Univalent Foundations of Mathematics'.[1] Here the authors mean 'foundation' in the sense of a framework or language for mathematical practice. They make a strong case that HoTT can indeed serve as a foundation in this sense, demonstrating how to characterize mathematical structures such as natural numbers, real numbers,

[1] Indeed, much of the interest in HoTT is due to the fact that it may be regarded as a 'programming language for mathematics', and it is formulated in a way that facilitates automated computer proof checking (HoTT Book [2013], p. 10).

and groups in the language of HoTT, and how to use it to formalize proofs in homotopy theory.

In this article, we grant for the sake of argument that HoTT is adequate as a framework for mathematics, and hence as a foundation in this restricted sense. However, philosophers often mean something stronger by 'foundation for mathematics'—they require a foundation to provide not just a language but also a conceptual and epistemological basis for mathematics, and moreover one that can be formulated without relying upon any other existing foundation.[2] If a system is to serve as a foundation for mathematics in this stronger sense, it must be possible to present it in a way that does not make reference to other parts of mathematics such as homotopy theory.

This article explains how to define, motivate, and think about HoTT in a way that is self-contained and does not depend upon other mathematics. So construed, HoTT is a candidate for being an autonomous foundation for mathematics; arguably, moreover, our interpretation improves on the standard one, as explained below. (Of course, this article does not present all the technical details of HoTT—for this, see (HoTT Book [2013]; Ladyman and Presnell [2014])—but it does introduce its essential features.) It is important to note that we do not alter the framework of HoTT, but rather reconstruct and interpret the formal theory given in the HoTT Book, often answering questions that its authors did not address rather than answering them differently. Thus, while this way of thinking about HoTT does not presuppose the homotopy interpretation, it also does not introduce anything that is incompatible with that interpretation.

In order to judge whether HoTT provides a foundation for mathematics, it is essential to be clear about what is meant by 'foundation'. There are many related notions in the folklore of the field, but not everyone in the field agrees about what is and is not required of a foundation of mathematics. It is precisely because this is not made make explicit that some disputes about the foundations of mathematics involve the protagonists talking past each other. In the next section we therefore list a number of questions of different kinds, pertaining to semantics, metaphysics, and epistemology, that a foundation for mathematics might be required to answer.[3]

Many foundational programmes give answers to some questions that make other questions particularly difficult. For example, if the proposed ontology of mathematics consists of abstract entities with no causal connection to the

[2] This is the notion of an 'autonomous' foundation similar to that of (Linnebo and Pettigrew [2011]) that we define below.

[3] We make no great claims for the originality of our characterization of foundation, but we are not aware of any similar explicit characterization in the literature. The most detailed treatment that we have found is the one due to Linnebo and Pettigrew ([2011]), but ours differs very significantly from theirs. We give further references to extant discussions of foundations in Footnote 8.

physical world, then this raises the question of why we should believe in such entities, and how we can come to have access to them at all (the 'access problem'; see Benacerraf and Putnam [1983], pp. 30–3). Ideally, a foundational programme should answer philosophical questions about mathematics without generating new, more difficult questions, and its answers should hang together with each other, with mathematical practice, and with our broader understanding of the world. For example, its epistemological claims should follow naturally from the ontology, without positing new, mysterious ways for the mind to gain access to knowledge of non-physical things. The interpretation of HoTT we propose in this article is intended to meet these desiderata, give answers to the questions of Section 2, avoid the access problem, and fit well with mathematical practice.

Our primary aim is not to convince the reader that HoTT can really do all this, nor that it is to be preferred as a foundation for mathematics over the usual foundation provided by ZFC set theory.[4] Nor is it to argue that mathematics must have a foundation that answers all the questions set out below.[5] Rather, our intention is to demonstrate (assuming HoTT's adequacy as a framework for mathematical practice) that, under our proposed interpretation, HoTT is worth the consideration of those who think mathematics should be given an autonomous philosophical foundation. Our secondary aim is to draw the attention of philosophers of mathematics to this new research programme that, as we argue below, offers an interestingly different way of thinking about mathematics. We explain the formal theory in some detail since existing presentations of HoTT are highly technical and not easily accessible. Hence, the reader should be able to learn about HoTT, its conceptual structure, and its philosophical dimensions.

The structure of the remainder of the article is as follows: Section 2 characterizes a foundation for mathematics. Section 3 introduces HoTT. Section 4 argues that the existing presentation of HoTT is not autonomous. Section 5 explains our interpretation of HoTT. Section 6 outlines our novel autonomous justification of the way identity is handled in HoTT (summarizing the argument of our companion paper; see Ladyman and Presnell [2015]). Section 7 shows how our interpretation of HoTT answers the questions of Section 2. Section 8 replies to some possible objections and Section 9

4 Although we do believe that it has a number of advantages, some of which are described in Section 9.1.

5 Some authors deny that any foundation for mathematics is necessary at all (see, for example, Putnam ([1967]). However, notice that some 'anti-foundationalists' who deny that we need and/or can have foundations of a particular kind may be happy with foundations in a weaker sense. For example, Awodey ([1996], [2014]) rejects a particular conception of foundationalism, but nonetheless he does think that answers must be given to at least some of the questions we set out below. Likewise, Shapiro ([1991]) rejects foundationalism while doing work in foundations more generally.

briefly concludes by raising the issue of whether HoTT under our interpretation ought to be adopted as a foundation for mathematics.[6]

2 What Is a Foundation for Mathematics?

In order to answer whether HoTT provides a foundation for mathematics, we must first establish what is meant by 'foundation'. Any discussion of the foundations of mathematics is complicated by the fact that advocates of different foundational programmes often have different ideas about what it is to provide a foundation, and what is taken to be required is not always made explicit. As mentioned above, one sense of foundation is that of a unifying language and conceptual framework, such as that of ZFC set theory. Another sense goes beyond this by adding particular definitions for mathematical entities in terms of that language (such as the Kuratowski definition of ordered pair, or the point-set definition of topological space in terms of sets).[7]

On the other hand, as also pointed out above, philosophers often seek a stronger kind of foundation for mathematics. Beyond merely giving a language for mathematics, a foundation in this sense involves providing a grounding for mathematics in pre-mathematical ideas, and answering semantic, metaphysical, epistemological, and/or methodological questions about mathematics.[8] In what follows, we will use the word 'foundation' to denote a foundation in this stronger sense unless otherwise indicated, reserving the work 'framework' for the weaker sense of foundation as language and conceptual framework. The notion of a framework is intended to distinguish the foundational work of many mathematicians, which is not concerned with the details of epistemology and metaphysics, from most foundational work in the philosophy of mathematics.

In this article we do not seek to answer the question of whether HoTT gives an adequate framework for the reconstruction of existing mathematics. One feature of HoTT that may be considered an obstruction to its providing such a framework is that (as we discuss in Section 3 below) the logic intrinsic to

[6] The main part of (HoTT Book [2013]) is about a version of the theory in which two further axioms are introduced called 'function extensionality' and 'univalence' (and in subsequent chapters, further extensions are considered). However, while these additions are of great interest, and univalence in particular is argued to be of philosophical significance in relation to mathematical structuralism (Awodey [2014]), they are beyond the scope of the present article, which focuses on the core of the theory and its interpretation without the addition of univalence or anything else.

[7] The distinction between these two positions essentially comes down to a question of how much of the groundwork of mathematics should be counted as part of the foundation, rather than as mathematical work built upon the foundational language. No philosophical issues relevant to the present discussion hang on this, and so we set aside this issue.

[8] See (Mayberry [1994]) for one explicit account of this sense of a foundation, for mathematics. For a discussion of different varieties of foundations, see (Shapiro [1991], Chapter 2). The classic discussion of the foundations of mathematics is (Benacerraf and Putnam [1983], Part 1). See (Shapiro [2005]) for recent reviews of the main foundational programmes.

HoTT is constructive rather than classical. If, to qualify as an adequate framework for mathematics, a system must provide the means to reconstruct all the theorems of classical mathematics as standardly stated, then any constructive foundation is ruled out (since, for example, the standard statement of the intermediate value theorem in analysis cannot be proved constructively). Of course, if HoTT were ruled out from being even a framework for mathematics in virtue of being constructive, then it could not further be a foundation for mathematics in the stronger sense sketched above, and the question posed in this article would be immediately answered in the negative. We assume for the remainder of the article that a constructive framework for mathematics is acceptable. (We take up the issue again briefly in Section 8.1.) Moreover, granting this assumption, we trust that the work done in the HoTT Book ([2013]) provides sufficient evidence for a cautious affirmative answer to the question of whether HoTT in particular is adequate as a framework for mathematics.[9] The question we are concerned with then is whether, on the assumption that HoTT is a candidate framework, it is also a candidate foundation for mathematics in the stronger philosophical sense.

To address this question, we must spell out in more detail the distinction between a framework and a foundation. Even those who are interested in foundations in the strong sense do not agree about their required features. There are therefore no necessary and sufficient conditions for a system to constitute a foundation. Rather, in the next subsection we offer a characterization of five components that a foundation might be expected to have, and articulate a series of questions that can be asked for each component. In Section 2.2 we introduce an important criterion, namely, that a foundation be autonomous, in the sense that it is not built upon some other foundational system. In the rest of the article, we use the analysis of this section to explore the foundational status of HoTT.

2.1 A characterization of a foundation for mathematics

There are five interrelated components to a foundation for mathematics and each generates a series of questions that a given putative foundation for mathematics might be expected to answer:

(1) A single framework in which to cast some or all of existing mathematics. This framework involves a mathematical language and theory that may be studied in its own right as well. This language may also

[9] See, in particular, the reconstructions of the natural numbers (HoTT [2013], Section 1.9), real numbers ([2013], Chapter 11), category theory ([2013], Chapter 9), and a model of ZFC set theory ([2013], Chapter 10).

be used in the everyday practice of mathematics, but this is not necessary.[10]

Questions: How are complex higher-level elements constructed or composed out of more basic ones? Is it built on a formal logic or not?[11] What role, if any, do axioms play?

(2) A semantics in the sense of an account of the basic concepts of the foundation, how the theoretical terms of (1) are to be understood, and an account of how the rules that are used to manipulate the concepts are to be understood.

Questions: What are the basic concepts and how are they related? Are statements expressed in this language to be understood as potentially having truth values and, if so, is the logic bivalent? How are the rules governing the elements of the language to be understood? What are the identity criteria for the elements of the theory? In particular, is the theory extensional or intensional?

(3) A metaphysics that spells out the ontological status of any entities posited in (2).

Questions: Does the metaphysics posit any objects at all?[12] Is the ontology (if any) to be understood as mind-dependent or mind-independent? What is the relationship between mathematical reality (if any) and physical reality?

(4) An epistemology in the sense of an account of how we are able to know the truths (if any) of mathematics, given (2) and (3) (which may also include an account of the applicability of mathematics), and a justification of the axioms and rules of the framework.

Questions: Given the answers to the above questions, what account, if any, do we give of mathematical knowledge? In particular, if basic entities are posited, how do we defend our claim to know about

[10] For example, most mathematicians do not work in the formal language of ZFC set theory, but this does not impugn its status as a framework for mathematics since definitions and theorems can (in principle) be translated into this language.

[11] Some foundational programmes presuppose a background formal logic as with, for example, axiomatic set theory, or with logicism, which reduces mathematics to logic. Others incorporate a logic informally by way of the rules that govern the mathematical theory, or advocate a particular logic for metaphysical and/or epistemological reasons, as with constructive logic and intuitionism. See (Shapiro [1991]) for a detailed investigation of the relationship between mathematics and logic.

[12] We may take a nominalist position, denying (or at least not asserting) that there are any mathematical objects (Field [1980]), in which case our answers to subsequent questions must be compatible with this.

them?[13] Given the role of proof in mathematical practice, what is the relationship between mathematical knowledge and proof? How are the rules justified, given their interpretation (if any)? If there are axioms, what is their epistemological status—for example, are they taken to be known, or are they taken to be merely hypothetical statements that form the antecedent of conditionals? What is the relationship between mathematical knowledge and knowledge of physical reality?

(5) A methodology for mathematical practice based on some or all of the above.

Questions: How is the foundation to be used in practice? In particular, how is it to be applied in the physical sciences?

Note that we do not claim that a putative foundation must be able to answer all these questions, nor must it definitively settle all the issues considered by philosophers of mathematics. However, to provide a foundation in the stronger sense, a system must at least speak to questions of semantics, metaphysics, and epistemology (even if only to say that no interpretation, metaphysics, and/or epistemology is to be given).

2.2 Autonomy

The answers given to the above questions may in some presentations be expressed in terms of some other foundational system. For example, an account of category theory might assume a background of ZFC set theory and define a category to be a set of objects and a set of morphisms satisfying certain rules. If this were the only way to present the system, then it would be hard to justify calling it a 'foundation' at all, since the concepts that were actually taken to be (conceptually, logically, or ontologically) fundamental would be those of the background mathematical system.

Following the terminology of Linnebo and Pettigrew ([2011]), we call a presentation of a putative foundation autonomous if and only if the answers given to the above questions depend only upon pre-mathematical ideas and principles. By 'pre-mathematical', we mean ideas and principles that can be explained and justified without recourse to mathematical ideas whose explanation or justification depends upon some other foundational system or some part of established mathematics. John Mayberry says: 'the primitive concepts of mathematics are those in terms of which all other mathematical concepts are ultimately defined, but which themselves are grasped directly, if grasped at

[13] For example, do we take the indispensability argument to justify a belief in mathematical objects (Colyvan [2001])?

all, without the mediation of a definition' ([1994], p. 18). There is no sharp distinction between the mathematical and the pre-mathematical, but there is a clear difference between, for example, a framework that uses the idea of membership of a collection and one that assumes ideas from advanced mathematics. For example, Linnebo and Pettigrew ([2011]) consider the charge that standard presentations of category theory fail to be autonomous because they rely upon ideas from set theory.

Of course, a given system may have multiple presentations, some autonomous and others non-autonomous. Moreover, these other ways to think about the system may have been the original motivation for devising it, and in practice may be more convenient or fruitful. The requirement of autonomy is only that there exists some way of arriving at the system via pre-mathematical considerations. Once this is established we are free to use whatever interpretation of the system is convenient for our day-to-day work.

As we noted above, mathematicians are often not at all concerned with the epistemology and metaphysics of mathematics, and so there is no reason to demand that a foundation, in the weaker sense of 'framework for mathematics', be autonomous. It is therefore entirely appropriate that these issues are not considered in the HoTT Book, and that the presentation in the HoTT Book weaves homotopy theory and type theory together from the beginning.

Hence, while in what follows we argue that the presentation of HoTT given by the authors of the HoTT Book is not autonomous, this should not be taken as criticism of their project in general, or of the HoTT Book in particular, since their aim in writing the book was to communicate their work to other mathematicians and computer scientists, and not to provide a foundation for mathematics in the philosophers' sense outlined above. It is because we think that HoTT is so promising as a framework for mathematics that we also think it is worth exploring the extent to which it is a foundation more broadly. Our work should thus be understood as complementary to the HoTT Book.

3 The Basic Features of Homotopy Type Theory

In this section, we summarize the basic principles of HoTT and consider the motivation for some of its rules.[14]

HoTT is built on the constructive intensional type theory of Martin-Löf ([1974]).[15] The framework of this type theory is a formal system consisting of

[14] This brief survey just introduces the main ideas and is not intended as a complete introduction or tutorial for HoTT. For a much more detailed exposition, see (HoTT Book [2013]; Ladyman and Presnell [2014]).

[15] Martin-Löf has his own interpretation and motivation for his theory that derives, in part, from his view of propositions that is briefly explained in Section 3.3. We explain how our view differs from his in Section 7.

types and tokens, in which each token belongs to exactly one type.[16] Tokens are always introduced and presented as tokens of some particular type: we can never have a token `x` whose type is unknown or uncertain. We write '$\mathtt{a} : \mathtt{A}$' to denote that token `a` is of type `A`. The theory is specified by giving rules for constructing new types and tokens from given types and tokens.

As an initial heuristic, it may be helpful to think of tokens as 'mathematical objects' and types as 'kinds of thing a mathematical object could be'.[17] For example, the type $\mathbb{N}$ of natural numbers can be defined, whose tokens are individual natural numbers.

In some literature on type theory, the word 'term' is used rather than 'token', where terms are thought of as syntactic entities, while types are semantic entities. Note that this is not what's going on here. Rather, the linguistic entities are expressions in a formal language that are taken to name tokens and types.

3.1 The rules

We begin with functions, which are defined by means of substitution into expressions as in lambda calculus (Church [1936]).[18] For any types `A` and `B`, a function of type $\mathtt{A} \rightarrow \mathtt{B}$ is defined by an expression, Φ, containing zero, one, or more instances of a variable x, such that when all instances of x are replaced by an expression naming a token of `A` (making sure to avoid re-using variable names in a way that changes the meaning of the expression), the resulting expression names a token of `B`. So, for example, given the expression '$(x + 5)/(2x - 3)$', when x is replaced by any expression that names a natural number, the resulting expression names a rational number. This therefore defines a function $[x \mapsto \Phi]$ (or in more traditional notation, $\lambda x.\Phi$) of type $\mathbb{N} \rightarrow \mathbb{Q}$, that is, a function that, when given any token of $\mathbb{N}$ as input, returns a token of $\mathbb{Q}$ as output.

Note that, in this approach, a function is therefore a computable procedure as in computer science, rather than an arbitrary pairing of input and output

[16] Terminology varies in the literature—in the HoTT Book, the words 'term', 'object', 'element', and 'point' are used interchangeably for what we are calling a 'token' of a type. However, each of these words carries a connotation that we wish to avoid: 'term' suggests something syntactic, 'object' begs the question about the semantics of the interpretation, 'element' suggests ideas from set theory, and 'point' suggests a spatial interpretation. We therefore prefer the word 'token', which is (we hope) metaphysically neutral, and which we intend to be understood as a term of art, independent of other usage.

[17] However, this is not the interpretation we will eventually settle upon, for reasons explained in Section 5.1.

[18] While lambda calculus is a sophisticated domain of study, the notion of substitution in an expression is a simple pre-mathematical one, familiar to anyone who, for example, is able to use pronouns in natural language. We therefore do not consider this to be an obstruction to the autonomous status of HoTT.

values as in classical mathematics founded in set theory. This is related to the constructive nature of the theory, as discussed in Section 8.1.

For any two types, $\mathtt{A}$ and $\mathtt{B}$, we have the function type $\mathtt{A}\to\mathtt{B}$ whose tokens are the functions defined as above. Note that, in this theory, functions are not of a fundamentally different character from any other token in the type theory. Thus whatever we may do with arbitrary tokens of arbitrary types, we may also do with functions. (In computer science terminology, functions are 'first class citizens'.)

Since not all types in the theory are function types, we must provide rules for how other types are defined as well. These rules consist of the following parts: a type former, one or more token constructors, an elimination rule, and possibly one or more computation rules. Before defining these in general, we first illustrate them with an example. The product of types is defined as follows:

- given two expressions, '$\mathtt{A}$' and '$\mathtt{B}$', that name types, the type former produces the expression '$\mathtt{A}\times\mathtt{B}$', which names another type, which we call the product of $\mathtt{A}$ and $\mathtt{B}$;
- given expression '$\mathtt{a}$' naming a token of type $\mathtt{A}$ and expression '$\mathtt{b}$' naming a token of type $\mathtt{B}$, the token constructor for the product type produces an expression '$(\mathtt{a},\mathtt{b})$' naming a token of type $\mathtt{A}\times\mathtt{B}$;
- the elimination rule for the product type is 'currying', which states that for any type $\mathtt{C}$, a function $\mathtt{f}:\mathtt{A}\times\mathtt{B}\to\mathtt{C}$ is given by a function $\mathtt{g}:\mathtt{A}\to(\mathtt{B}\to\mathtt{C})$ that takes tokens $\mathtt{a}:\mathtt{A}$ as input and gives functions $\mathtt{g}_\mathtt{a}:\mathtt{B}\to\mathtt{C}$ as output.
- the computation rule for the product type states that for any expression '$(\mathtt{a},\ \mathtt{b})$' naming a token of type $\mathtt{A}\times\mathtt{B}$, the expression '$\mathtt{f}((\mathtt{a},\mathtt{b}))$', names the same token of type $\mathtt{C}$ as the expression '$\mathtt{g}_\mathtt{a}(\mathtt{b})$'.

We write the computation rule as $\mathtt{f}((\mathtt{a},\mathtt{b})) :\equiv \mathtt{g}_\mathtt{a}(\mathtt{b})$, where '$exp_1 \equiv exp_2$' means 'expressions exp_1 and exp_2 name the same token or type', and '$exp_1 :\equiv exp_2$' means 'by definition, expression exp_1 names the same token or type as exp_2'. Thus when we have $exp_1 \equiv exp_2$, any instance of exp_1 can be replaced in any expression by exp_2 in any context whatsoever without changing the meaning.

While the above example has been explained by reference to expressions of the formal language, it is generally more convenient to talk in terms of functions. To define a type we require functions as follows:

- a type former that when given suitable types (and perhaps tokens of types) as inputs produces the new type as output;
- one or more token constructors that output tokens of the new type, given tokens of any required input types;

- one or more elimination rules that provide functions that take tokens of the new type as their inputs;
- computation rules that define the behaviour of the functions given by the elimination rules by specifying their outputs when given particular inputs.

This is the general framework for defining types.

3.2 The basic ways to construct types

The basic ways to construct types in HoTT (with one exception, discussed below) are as follows[19]:

- Given any two types, $\mathtt{A}$ and $\mathtt{B}$, we can form function type $\mathtt{A}\to\mathtt{B}$. Tokens of this type are functions that take a token of $\mathtt{A}$ as input and return a token of $\mathtt{B}$ as output (defined as described above).
- Given any two types, $\mathtt{A}$ and $\mathtt{B}$, we can form product type $\mathtt{A}\times\mathtt{B}$. A token of $\mathtt{A}\times\mathtt{B}$ is a pair, $(\mathtt{a},\mathtt{b})$, where $\mathtt{a} : \mathtt{A}$ and $\mathtt{b} : \mathtt{B}$. The elimination rule is given by 'currying', which for any type $\mathtt{C}$ produces a function of type $\mathtt{A}\times\mathtt{B}\to\mathtt{C}$ from a function of type $\mathtt{A}\to(\mathtt{B}\to\mathtt{C})$.
- Given any two types, $\mathtt{A}$ and $\mathtt{B}$, we can form coproduct type $\mathtt{A}+\mathtt{B}$. The token constructors are functions $\mathtt{inl} : \mathtt{A}\to\mathtt{A}+\mathtt{B}$ and $\mathtt{inr} : \mathtt{B}\to\mathtt{A}+\mathtt{B}$. The elimination rule states that if we have functions $\mathtt{g_l} : \mathtt{A}\to\mathtt{C}$ and $\mathtt{g_r} : \mathtt{B}\to\mathtt{C}$ (for some type $\mathtt{C}$), then we can form a function $\mathtt{g} : \mathtt{A}+\mathtt{B}\to\mathtt{C}$ that applies either $\mathtt{g_l}$ or $\mathtt{g_r}$ to its input as appropriate.
- There is a unit type $\mathtt{1}$, which has a single token constructor producing a token $* : \mathtt{1}$. The elimination rule states that to define a function $\mathtt{f} : \mathtt{1}\to\mathtt{C}$ for any given output type $\mathtt{C}$, we need a single token $\mathtt{c} : \mathtt{C}$ to serve as the value of $\mathtt{f}(*)$.
- There is a zero type $\mathtt{0}$, which has no token constructors. The elimination rule states that for any given output type $\mathtt{C}$, there is a function $!_{\mathtt{C}} : \mathtt{0}\to\mathtt{C}$.
- Recall that a token of $\mathtt{A}\times\mathtt{B}$ is a pair $(\mathtt{a},\mathtt{b})$ whose second component is of type $\mathtt{B}$. We can relax this, allowing the second component of a pair to be of a type that depends upon the first component. This gives the dependent pair type. Given a type, $\mathtt{A}$, and a family of types, $\mathtt{P}(\mathtt{a})$, indexed by $\mathtt{A}$, the type former produces the dependent pair type $\sum_{\mathtt{x}:\mathtt{A}}\mathtt{P(x)}$. Tokens of this type, called 'dependent pairs', are pairs $(\mathtt{y},\mathtt{q})$, where $\mathtt{y} : \mathtt{A}$ and $\mathtt{q} : \mathtt{P}(\mathtt{y})$.
- We can likewise generalize the function type to relax the restriction that the output of a function is always of a particular fixed type, regardless of what token of the input type it is given. Given a type $\mathtt{A}$ and a family of

[19] The computation rules, which essentially state that the constructors and eliminators behave as expected, are not spelled out here. For these and other details see (HoTT Book [2013], Chapter 1).

types $\mathtt{P}(\mathtt{a})$ indexed by $\mathtt{A}$, the type former produces dependent function type $\prod_{\mathtt{x:A}}\mathtt{P(x)}$. Tokens of this type, called 'dependent functions', are functions that return a token of type $\mathtt{P}(\mathtt{a})$ when given $\mathtt{a : A}$ as input.

There is an important component of HoTT missing from the above, namely, identity types, which we introduce in Section 3.4. Call the system sketched above HoTT$^-$ to distinguish it from full HoTT.[20] Before introducing identity types, we first discuss the interpretation and justification of the rules of HoTT$^-$.

3.3 Types as propositions and propositions as types

The above type definitions can be motivated and justified via an interpretation of types as propositions. It was noted by Curry and Howard (Curry [1934]; Howard [1980]) that there is a correspondence between computations in type theory and natural deduction. For example, if we have a function, $\mathtt{f}$, of type $\mathtt{A}\to\mathtt{B}$ and token $\mathtt{x}$ of type $\mathtt{A}$, then applying $\mathtt{f}$ to $\mathtt{x}$ gives, by definition, a token of type $\mathtt{B}$. This is formally parallel to the rule of *modus ponens*: if proposition $A \Rightarrow B$ is true and proposition A is true, then it follows that proposition B is also true. The Curry–Howard correspondence (or 'equivalence' or 'isomorphism') extends this to other logical operations.

Per Martin-Löf's 'meaning interpretation' of propositions (Martin-Löf [1996]) makes this more than a merely formal parallel. In his system, every proposition may be represented by a collection of things that stand as 'evidence' or 'proofs' of that proposition, and logical reasoning is then simply the manipulation of pieces of evidence according to rules. The Curry–Howard correspondence is understood as saying that to each proposition there corresponds a type, the tokens of which are proofs (or, as we say, 'certificates') of that proposition.[21] Thus we can assert that a proposition is true just in case we have a token of the corresponding type (the type is said to be 'inhabited'). This leads to the Brouwer–Heyting–Kolmogorov (BHK) interpretation of constructive logic (Troelstra [2011], p. 161), in which a certificate to a compound proposition can be understood in terms of certificates to its constituent propositions. The basic rules for constructing new types from old ones, given the above, then correspond to logical operations such as conjunction and disjunction:

[20] Recall from Footnote 6 that by 'HoTT' we mean the core of the theory without extensions such as the univalence axiom.

[21] Again, terminology varies in the literature. Martin-Löf ([1996]) uses 'proofs', which we avoid for reasons explained in Section 7.2. In the HoTT Book, 'witness' and 'evidence' are used. We prefer 'certificate' over the alternatives, since it makes for a better analogy with everyday usage: whereas evidence indicates the truth of a proposition, and a witness observes the truth, a legal document such as a contract or a marriage certificate makes something true.

- The implication relation $A \Rightarrow B$ between propositions is represented by function type $\mathtt{A}\rightarrow\mathtt{B}$. The application of a function to an input corresponds to *modus ponens*, as described above.
- The conjunction of two propositions is represented by the product of the corresponding types, $\mathtt{A} \times \mathtt{B}$. The token construction rule corresponds to the rule of $\wedge$-introduction: from a certificate to $\mathtt{A}$ and a certificate to $\mathtt{B}$, we can produce a certificate to the conjunction. Similarly, the elimination and computation rules correspond to $\wedge$-elimination.
- The disjunction of two propositions is represented by the coproduct of the corresponding types, $\mathtt{A} + \mathtt{B}$. The token construction rules correspond to the $\vee$-introduction rule of constructive logic: to produce a certificate to $A \vee B$ we must have either a certificate to A or a certificate to B. The elimination rule corresponds to $\vee$-elimination: given certificates to $A \Rightarrow C$ and $B \Rightarrow C$ then we can produce a certificate to $(A \vee B) \Rightarrow C$.
- The zero type corresponds to a contradictory proposition, hence no certificate to this proposition can be (directly) produced.[22] The elimination rule corresponds to the law of explosion: from a contradictory proposition any consequence follows.
- Given a proposition, A, with corresponding type $\mathtt{A}$, the negation of A corresponds to type $\mathtt{A}\rightarrow\mathtt{0}$. A token of $\mathtt{A}\rightarrow\mathtt{0}$ is therefore a function that, if it were given a token of $\mathtt{A}$, would produce a token of $\mathtt{0}$, which would certify the truth of a contradiction.
- A family of types $\mathtt{P}(\mathtt{a})$ indexed by $\mathtt{A}$ corresponds to a predicate on $\mathtt{A}$. For a particular token, $\mathtt{x} : \mathtt{A}$, type $\mathtt{P}(\mathtt{x})$ corresponds to the proposition that $\mathtt{x}$ satisfies this predicate.
- An existentially quantified proposition corresponds to the dependent pair type $\sum_{\mathtt{a}:\mathtt{A}}\mathtt{P}(\mathtt{a})$. A token of this type is a pair $(\mathtt{x}, \mathtt{p})$ whose first component is token $\mathtt{x}$ of $\mathtt{A}$ and whose second component, $\mathtt{p} : \mathtt{P}(\mathtt{x})$, is a certificate to the fact that this $\mathtt{x}$ satisfies the predicate. Such a pair is (in constructive logic) a certificate to an existentially quantified proposition.
- A universally quantified proposition corresponds to the dependent function type, $\prod_{\mathtt{x}:\mathtt{A}}\mathtt{P}(\mathtt{x})$. A token of this type is a dependent function that, when given token $\mathtt{x}$ of type $\mathtt{A}$, returns a token of type $\mathtt{P}(\mathtt{x})$. Such a dependent function is a certificate to the fact that all tokens of type $\mathtt{A}$ satisfy the predicate.

[22] Consistency of the system then consists in the claim that no token of $\mathtt{0}$ can be produced by any means.

Note that the logic that is encoded into the definition of the token constructors and elimination rules is constructive. In particular, to produce a certificate to a disjunction, we must have a certificate to one or other of the disjuncts, and to produce a certificate to the negation of a proposition, we must demonstrate how to derive contradiction from that proposition. Thus the law of excluded middle (LEM) and the law of double megation elimination do not hold as laws of logic in this system, since it is not in general possible to produce a certificate to $A \vee \neg A$ or $\neg\neg A \Rightarrow A$ for arbitrary propositions, A. (This issue is discussed further in Section 8.1.)

The interpretation of types as propositions, in conjunction with the BHK interpretation of constructive logic, leads directly to the rules of HoTT^{-}. Moreover, this interpretation can be motivated and explained without any recourse to existing mathematics such as set theory.[23] The rules of HoTT^{-} can therefore be part of an autonomous foundation for mathematics.

Applying these type formation rules, we can build up many complex types corresponding to complex mathematical propositions. A proof of such a proposition from some given premises consists of a sequence of applications of token constructors and elimination rules, beginning with the given certificates to the premises and ending with a certificate to the proposition.

The above rules give us ways of manipulating types, but provide almost no content with which to begin working. In practice, we must introduce additional definitions to define the objects of study: natural numbers, real numbers, and so on.[24]

Unlike the axioms of ZFC set theory, in which the existence of the objects of study (that is, pure sets) follows immediately from the axioms of the theory, in HoTT the basic rules of the theory do not assert the existence of any particular types or tokens (beyond a few very simple ones related to the zero and unit types). This allows HoTT to be given an ontologically minimal interpretation, as discussed in Section 7.

3.4 Identity

The above definitions are not sufficient to provide a language for mathematics, since they do not provide an adequate way to assert that two things are equal. The statement that two things are equal is a proposition, and so, in line

[23] In the case of the quantified propositions, we must, of course, interpret 'domains of quantification' correctly: not as sets whose elements are members, but as types whose elements are tokens.

[24] For example, the type $\mathbb{N}$, whose tokens are natural numbers, has two token constructors $\mathtt{z} : \mathtt{1} \to \mathbb{N}$, which gives the zero element, and $\mathtt{s} : \mathbb{N} \to \mathbb{N}$, the successor. We can therefore produce tokens $\mathtt{z}(*) : \mathbb{N}$ (which we write as '0' for convenience), and $\mathtt{s}(0) : \mathbb{N}$, and $\mathtt{s}(\mathtt{s}(0)) : \mathbb{N}$, and so on. The elimination rule then tells us how to construct a function of type $\mathbb{N} \to \mathtt{C}$, for arbitrary type $\mathtt{C}$, given a token $\mathtt{c_z} : \mathtt{C}$ and a function $\mathtt{c_s} : \mathtt{C} \to \mathtt{C}$. For more details, see (HoTT Book [2013], Section 1.9).

with the correspondence between propositions and types discussed above, we should expect to have a corresponding type in the theory. However, the judgemental or external equality relation $\equiv$ introduced in Section 3.1 does not name a type in the theory, but rather says something about expressions of the theory, namely, that they name the same token or type. Such external judgements cannot be combined with other elements of the language to produce more complex statements; for example, the trivial number-theoretic statement $\forall \mathtt{m}, \mathtt{n} : \mathbb{N}, (\mathtt{m} = \mathtt{n}) \Rightarrow (\mathtt{s}(\mathtt{m}) = \mathtt{s}(\mathtt{n}))$ cannot be expressed using judgemental equality. Rather, what is needed is a new way of forming types that correspond to the proposition that two particular things are equal. In the remainder of this section, we define these identity types.

The type former for identity types gives, for each type $\mathtt{C}$ and any two tokens $\mathtt{a} : \mathtt{C}$ and $\mathtt{b} : \mathtt{C}$, type $\mathtt{Id}_{\mathtt{C}}(\mathtt{a}, \mathtt{b})$ that corresponds to the proposition that $\mathtt{a}$ and $\mathtt{b}$ are equal. A token of the type (that is, a certificate to the proposition) is called an 'identification' of $\mathtt{a}$ and $\mathtt{b}$.

The token constructor for identity types produces, for any type $\mathtt{C}$ and any token $\mathtt{a} : \mathtt{C}$, a certificate to the fact that $\mathtt{a}$ is identical to itself. These tokens, $\mathtt{refl}_{\mathtt{a}} : \mathtt{Id}_{\mathtt{C}}(\mathtt{a}, \mathtt{a})$ (where 'refl' is short for 'reflexivity of identity'), are called 'trivial self-identifications'.

This much is easily justified: we should be able to form the proposition '$a = b$' for any two tokens of the same type (but not for two tokens of distinct types), but in the absence of further assumptions, we should only be able to prove such a proposition (that is, construct a token of the type) in the trivial case '$a = a$' (and we should always be able to prove this).

The relation between judgemental equality and identity is asymmetric in the theory. For any type $\mathtt{C}$ and any tokens $\mathtt{a} : \mathtt{C}$ and $\mathtt{b} : \mathtt{C}$, given a judgemental equality $\mathtt{a} \equiv \mathtt{b}$, we can derive a token of identity type $\mathtt{Id}_{\mathtt{C}}(\mathtt{a}, \mathtt{b})$. However, the converse 'reflection rule' does not hold: given a token of $\mathtt{Id}_{\mathtt{C}}(\mathtt{a}, \mathtt{b})$ we cannot conclude that a judgemental equality $\mathtt{a} \equiv \mathtt{b}$ holds. (Indeed, if such a rule obtained, then all identity types would be trivial and of no interest, since all identifications would be judgementally equal to trivial self-identifications; see HoTT Book [2013], Exercise 2.14.) The failure of this reflection rule is the definition of 'intensionality' for a type theory (Martin-Löf [1974]).

Note that although trivial self-identifications of the form $\mathtt{refl}_{\mathtt{a}}$ are the only identifications that can be freely constructed, we cannot prove (in the absence of the reflection rule) that these are the only identifications that exist. That is, although without additional premises we cannot construct non-trivial identifications, we also cannot prove that none exist.[25] (This is a common

[25] Indeed, it is this fact that allows identity types to take on the rich and interesting structure of '∞-groupoids' that allows for a more tractable approach to problems previously treated in n-category theory, and is part of what makes the theory so mathematically interesting, and allows the homotopy interpretation discussed in Section 3.5 (see HoTT Book [2013], Chapter 2).

situation in constructive logic: consider, for example, the non-zero infinitesimals in synthetic differential geometry; Bell [2008].)

3.4.1 Path induction

To complete the definition of the identity type, we must also give its elimination rule, which is called 'path induction'. Just as mathematical induction on $\mathbb{N}$ allows us to prove that some condition holds of all natural numbers without considering each natural number individually, likewise path induction allows us to prove that some condition holds of all identifications between tokens of some type. In an induction on $\mathbb{N}$, we first prove that the condition holds of 0 (the base case), then show that if it holds of some n, then it also holds of $n+1$ (the inductive step). A proof using path induction is even simpler than this, since only a base case needs to be proved—the inductive step holds as a theorem for all properties that we might consider. That is, the principle of path induction states that for any type $\mathtt{C}$ and any property $\mathtt{P}$ that can be asserted of identifications between tokens of $\mathtt{C}$, if we can prove that $\mathtt{P}$ holds of all trivial self-identifications $\mathtt{refl}_{\mathtt{a}}$ for all $\mathtt{a} : \mathtt{C}$, then it holds of all identifications in C. (A formal statement of path induction is given in Section 6.)

Path induction is a very powerful tool, since in many cases it makes proofs about identifications very simple. However, its interpretation and justification are not so straightforward. One natural way to interpret it is as a statement that all identifications are trivial self-identifications. However, as noted above, this is not something that can be proved in HoTT and, moreover, this is not the intended interpretation, since it eliminates all the structure of identity types that is of particular interest in HoTT ([2013], Section 1.12).

The propositions as types interpretation set out in the previous section does not extend to give an account of identifications between tokens. Thinking of tokens as certificates to propositions gives no motivation for path induction. Instead, the HoTT Book explains and justifies path induction via the homotopy interpretation of HoTT ([2013], Section 1.12), which we sketch below.

3.5 The homotopy interpretation

3.5.1 A sketch of homotopy theory

One of the major innovations of HoTT is the alternative interpretation of types and tokens it provides using ideas from homotopy theory, which was

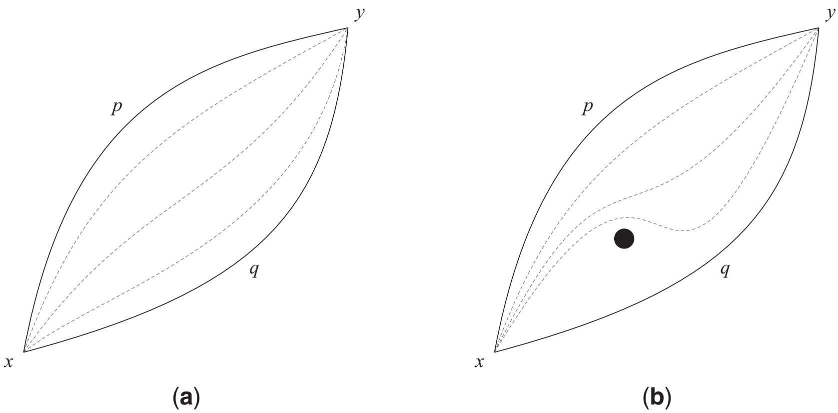

Figure 1. (a) p and q are paths between points x and y in the Euclidean plane. The dashed lines indicate some of the intermediate paths in a homotopy between p and q. (b) In a space with a hole or obstruction between paths p and q, continuous distortions of p cannot jump over the obstruction and so p and q, are not homotopic.

developed by Awodey and Warren ([2009]). In this section we give a very brief survey of homotopy theory and the main points of this interpretation.[26]

Homotopy theory is the study of spaces and functions between spaces up to continuous distortion. That is, if there is a continuous deformation that transforms one topological space into another, or one continuous function into another, then in homotopy theory they are regarded as equivalent.

More precisely, two continuous functions, $f, g : A \to B$, between topological spaces are homotopic (written $f \sim g$) if there is a continuous function $h : [0, 1] \times A \to B$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in A$. Such a function is called a 'homotopy' between f and g, and provides a continuous interpolation between the two functions.

Given two points, x and y, in space X, a path between them is a continuous function, $\gamma : [0, 1] \to X$, with $\gamma(0) = x$ and $\gamma(1) = y$. So, for example, any two paths between any two points, x and y, in the Euclidean plane are homotopic to one another, because they can be continuously deformed into one another (Figure 1(a)). However, in a space with a hole in it (such as an annulus), there can be paths between two points that are not homotopic, since a path going one way around the hole cannot be continuously deformed into a path going the other way around the hole (Figure 1(b)).[27]

[26] We use the familiar language of set theory and classical logic for this presentation, rather than casting everything into the language of HoTT itself.

[27] These ideas have important applications in physics. For example, the celebrated Aharonov–Bohm effect depends upon the fact that the space in which the electrons move is punctured by a 'hole', namely, the solenoid around which the electrons pass (Aharonov and Bohm [1959]; Healey [1997]).

This relation between functions leads naturally to an equivalence relation, called 'homotopy equivalence', between topological spaces.[28] Homotopy theory does not distinguish between functions that are homotopic, or between spaces that are homotopy equivalent. Thus in homotopy theory, facts about spaces and functions can only be specified up to continuous distortions, and only facts that are preserved by all such distortions are well defined.

3.5.2 The homotopy interpretation of HoTT

For reasons of space and technicality, we cannot give a complete explanation of the homotopy interpretation of HoTT, but in this section we give a brief overview.[29]

Awodey and Warren ([2009]) introduced an interpretation of the language described above, in which the types of the theory are interpreted as spaces (as thought of in homotopy theory) and the basic operations on types are interpreted as operations on spaces. It is then natural to interpret tokens of a type as points in the corresponding space.

A path between points x and y in space X may itself be regarded as a homotopy between the constant functions $k_x : \bullet \to X$ and $k_y : \bullet \to X$, which map the one-point space $\bullet$ to the points x and y, respectively. Thus it is natural, on the homotopy interpretation, to interpret identifications between tokens as paths between the corresponding points. The constant path at x (that maps every element of $[0, 1]$ to x) corresponds to the trivial self-identification of x with itself.

This also provides an important part of the explanation of the principle of path induction given in the HoTT Book ([2013], Section 1.12). Any path, p, between x and y is homotopic to a constant path at x: keeping one end fixed at x, the path can be continuously retracted along its length. Thus any property of the constant path at x that respects homotopy must be shared by p. Thus to show that every path starting at x has such a property, it suffices to show that the constant path at x has that property. More generally, to show that all paths have such a property, it suffices to show that all constant paths have the property. This is the homotopy-theoretic counterpart to the principle of path induction.

4 Autonomy of the Standard Presentation?

The presentation of HoTT given in the previous section (which sketches that of the HoTT Book, henceforth called the 'standard presentation' of HoTT)

[28] Specifically, two spaces, X and Y, are homotopy equivalent if there are continuous maps $f : X \to Y$ and $f' : Y \to X$ such that $f' \circ f \sim idX$ and $f \circ f' \sim idY$. For example, a disc and a single point are homotopy equivalent. Likewise, a circle, an annulus, and a (solid) torus are all homotopy equivalent. But the circle is not homotopy equivalent to the single point.

[29] For more details, see (HoTT Book [2013]).

allows us to answer many of the questions posed in Section 2. Briefly, the framework of the theory is given by the language outlined in Section 3, the semantics is given by the homotopy interpretation sketched in Section 3.5, and no explicit metaphysics or epistemology is given. (As well as presenting our interpretation, Section 7 gives more details of the standard presentation.)

Since the semantics of this presentation explicitly depends upon the terminology and concepts of homotopy theory, the standard presentation is manifestly not autonomous. The interpretation of types, tokens, and identifications requires an understanding of spaces, points, and paths that is not pre-mathematical, but rather derives from the mathematics of homotopy theory.

This is not an accidental feature of the standard presentation: throughout the HoTT Book, the terminology and concepts of homotopy form a central part of the presentation. The distinction between the type theory per se and the homotopy interpretation of it is not made, and the authors switch freely between the terminology of the two. As the authors themselves say, 'Homotopy Type Theory (HoTT) interprets type theory from a homotopical perspective' (HoTT Book [2013], p. 3) and the interpretation of identifications as paths is described as 'the key new idea of the homotopy interpretation' ([2013], p. 5). Spaces, points, and paths are given centre stage when explaining new ideas; for example, the principle of path induction, which is a fundamental part of the theory, is justified using homotopical reasoning as sketched in Section 3.5 above ([2013], Section 1.12).[30]

Of course, the fact that the homotopy interpretation is used throughout the presentation does not settle its exact role in the theory itself, and it does not follow that the fundamentals of the theory inherently depend upon homotopy theory, or that an alternative presentation cannot be given. Any such presentation, if it is to be autonomous, must address the following two questions:

(1) How can we understand types and tokens without the homotopy interpretation?

(2) How are the rules of HoTT to be justified on the basis of that understanding? In particular, how is path induction to be justified without reference to the homotopy interpretation?

5 The Interpretation of Tokens and Types

In the standard presentation of HoTT types are to be interpreted either as propositions or as spaces (more specifically, as homotopy types). As we noted

[30] None of this is intended as a criticism of the HoTT Book or its authors. The goal of the authors of the HoTT Book is not to present HoTT as an autonomous foundation in the sense considered here, but to introduce it as a subject of study for mathematicians. Since much of the innovative use of HoTT is derived from the homotopy interpretation, it is completely natural to want the reader to become familiar with that interpretation.

in Section 3.4, the interpretation of types as propositions does not give a justification of path induction or other features of identity types. The homotopy interpretation does give a justification of path induction, and pervades the way of thinking about identity presented in the HoTT Book, but clearly depends upon the sophisticated machinery of homotopy theory and is therefore incompatible with the autonomy of the foundation.

One response to this problem is simply to drop the interpretation of types from our account of HoTT altogether, adopt a 'null semantics', and thus take an extreme formalist approach.[31] That is, one could say that the foundation provided by HoTT consists only of a formal language with rules for the manipulation of symbols, but that the symbols should not be interpreted as having any meaning at all. This is clearly compatible with the foundation's being autonomous, and furthermore eliminates any need to explain or justify any metaphysical assumptions, since none are made. We will not rehearse the objections to formalism here, but we do show that a richer foundational account is available: in Section 5.2 we give a semantics that is compatible with the intensional and constructive nature of the theory.

We might alternatively try to preserve as much of the standard presentation as possible by giving an account of spaces that contains the features needed to support this interpretation, but which is grounded in pre-mathematical intuitions rather than homotopy theory. That is, by inspection of our intuitive concept of what a space is, we might hope to recover those features of homotopy theory that are used in the homotopy interpretation of HoTT without needing to set up all the mathematical machinery that is standardly used to define it. However, it is not at all clear that such an argument—that the required features of homotopy theory are all present in our intuitive notion of space—can be defended, and we do not pursue this approach further in the present article.

In the next subsection, we will argue why tokens should not be interpreted as mathematical objects, before introducing our interpretation of types and tokens as concepts in Section 5.2.

5.1 Tokens as mathematical objects?

As noted in Section 3, a natural way to interpret tokens and types is to say that tokens correspond to mathematical objects and types correspond to kinds of mathematical objects, so that `a : A` means that object `a` is of kind `A`. However, there are problems with this interpretation.

First, it commits us to a strong kind of Platonism, since we can only adopt this interpretation if we believe that there are mathematical objects; we would

[31] Formalism in general comes in different forms, see (Detlefsen [2005]).

then have to give an account of the metaphysics of these objects and our epistemic access to them.

Second, this interpretation does not naturally accord with the intensional treatment of types. In particular, while a description of an object may omit some of its properties and leave it partially specified, any actual object must possess all the relevant properties, whether they are specified in its description or not. Thus, for example, any actual triangle must be either equilateral, isosceles, or scalene; an abstract triangle that is merely triangular and not, more specifically, also equilateral, isosceles, or scalene is therefore problematic.

Third, whereas complex types are constructed from simpler types by application of the type formers, there is no corresponding sense in which complex kinds are composed from simpler kinds. The definition or description of a kind is, of course, constructed by a logical assembly of simpler notions, but the kind itself is not.

Fourth, this interpretation does not explain why we take each token to belong to exactly one type. We would normally say that a given mathematical object can be of multiple kinds, just as a physical object can be of multiple kinds—for example, a particular dog is of the kinds 'dog', 'mammal', and 'pet', and a particular number may be of the kinds 'natural number', 'odd number', and 'prime number'. The restriction to a single type for each token is, on this interpretation, both unexplained and unnatural.

Finally, this interpretation arguably gets the relationship between tokens and types backwards. When we classify physical objects into kinds, we take the objects themselves to be primary and the kinds to be secondary, dependent upon the objects. However, in the type theory sketched above, the order is the other way around: token constructors can only be specified once the type itself has been defined. To adopt this interpretation, we would therefore need to explain this reversal.

Hence, while this approach is initially appealing, it does not accord well with the features of the type theory to be interpreted.

5.2 Tokens and types as concepts

In this section, we present an alternative account of tokens and types as concepts. This account originates in the observation that we can have mathematical concepts, whether or not there are any mathematical objects. For example, we have the idea of natural numbers whether or not we believe that there are any such things. We can therefore say that, rather than picking out mathematical objects, tokens correspond to specific mathematical concepts; and rather than denoting kinds of mathematical objects, types correspond to general mathematical concepts. Thus $\mathtt{a} : \mathtt{A}$ would mean that specific concept $\mathtt{a}$ falls under, or is an instance of, general concept $\mathtt{A}$. For example, the natural number three is a token of the type 'natural number', being a specific instance of the general kind. Similarly, three-dimensional Euclidean space is a token of the type 'metric space'.

It is clear that concepts are pre-mathematical in the sense of Section 2.2. That is, without having learned any mathematics someone can understand the idea of a concept, and at no point does the development of this understanding require ideas from mathematics. Thus this interpretation is suitable for an autonomous foundation for mathematics.

To develop this interpretation we must say a few things about concepts. Of course, we do not here give a complete account of concepts, but we highlight some features that are relevant to the present discussion:

(1) The existence of concepts does not depend upon the existence of specific objects that they may represent. So, for example, the existence of the concept of 'unicorn' does not require that there be any unicorns in the world, and having the concept does not commit us to believing that any exist. Indeed, it is necessary to have the concept 'unicorn' even to be able to frame the denial that such things actually exist.

(2) Concepts can be of the concrete or abstract, and can be specific relative to a more general concept. For example, 'dog' is a specific concept relative to the general concept 'mammal'. Both are concrete concepts, as opposed to 'justice', which is an abstract concept. Specific concepts can stand in relation to general concepts just as specific objects stand in relation to kinds; for example, courage is a virtue. We take it that for any specific object or general kind there is a corresponding concept. Thus a foundation that takes concepts as primitive entities is compatible with Platonism about mathematics.[32]

(3) While concepts only have effects via mental activity, they may or may not depend for their existence on mental activity—that is, one might hold that a concept does not spring into existence when the definition is first stated. For example, the concept 'triangle' may exist even while no-one is presently thinking about triangles, and the concept 'elliptic curve' may have existed even before anyone first conceived of elliptic curves. Thus a foundation that takes concepts as primitive entities is not thereby committed to saying that mathematics is subjective or intersubjective rather than being concerned with an objective mind-independent subject matter.[33]

[32] It is an open question whether the posited objects would then be redundant in mathematical practice, and we do not address this here.

[33] See Section 8.2 for further discussion of this issue.

(4) Concepts are intensional: they correspond (roughly) to descriptions rather than to extensional collections. Hence (to use two famous examples), 'the morning star' and 'the evening star' are two distinct specific intensions, although they have the same extension; and 'human' and 'featherless biped' are distinct general intensions although they have the same extension.[34] We can have empty concepts, even necessarily empty concepts, and indeed multiple distinct empty concepts.

(5) Concepts can be composed and refined to form more complex concepts.

(6) Being intensional, concepts can be partially specified. So, for example, whereas any specific triangle must be equilateral, isosceles, or scalene, the concept 'triangle' (characterized, for example, as 'plane figure with three sides') does not require any particular relationship between side lengths. Concepts can therefore be just as precisely specified as required, in a way that objects cannot.

Let's consider again the problems that faced the interpretation of tokens and types as objects and kinds. Since the existence of concepts does not depend upon the existence of entities, we are not committed to mathematical Platonism. The intensional treatment of types fits perfectly with an interpretation of types as concepts, since concepts are themselves intensional. Complex concepts may be composed from simpler ones, just as complex descriptions can be composed.

However, the interpretation of tokens and types as specific and general concepts, respectively, does not answer the remaining two issues, namely, why each token belongs to exactly one type, and what the dependence relationship between tokens and types is. Therefore, our account must be slightly refined to address these points.

Since concepts are intensional, they can be specified in ways that emphasize some particular feature or aspect. We therefore interpret a token not simply as the concept of some specific mathematical thing, but rather as the concept of a specific thing *qua* instance of a general concept. This explains why each token belongs to exactly one type, and gets the order of dependence right: we must have the general concept first before we can have the concept of some specific thing *qua* instance of that general concept.

How does this interpretation accord with the treatment of identity in HoTT? What does it mean to identify two specific concepts (*qua* instance of some general concept)? Consider, for example, two triangles at two different locations in the Euclidean plane. Since they are at different locations, they are

[34] Moreover, mathematical concepts are hyperintensional, since even intensions that are necessarily coextensional, such as 'equiangular triangle' and 'equilinear triangle', are distinct.

clearly distinct—two triangles in the Euclidean plane are identical just if they have precisely the same three vertices. However, if we abstract away their locations, the criterion of identity becomes congruence (that is, having the same angles and side lengths). If we further abstract away considerations of size, then the criterion of identity becomes similarity (that is, having the same angles). Thus two triangles in the Euclidean plane may be numerically distinct whilst being identical *qua* instance of some general concept (whereby some of their specific features are abstracted away).

By allowing us naturally to abstract away some features and retain others, the interpretation as concepts fits well with the treatment of identity.[35] Moreover, this gives an illustration as to how a pair of tokens may be identified in multiple distinct ways. For example, two triangles can be shown to be similar by giving a correspondence between their angles or, alternatively, by showing that their side-lengths are the same.

Finally, we must address the relation between concepts and propositions. The rules of HoTT^{-} were justified via the BHK interpretation of constructive logic, but typically we would consider the entities manipulated in logical reasoning to be propositions. In the interpretation proposed here, we take these entities (namely, the types of the theory) to be general concepts. However, this change in perspective is not problematic. The general concepts considered in mathematics are those that can be defined precisely in some mathematical language. Of course, we may initially come to some mathematical concept via informal means—for example, initially thinking of real numbers as the lengths of geometric lines. But, in modern mathematics at least, such concepts must be given rigorous and precise expression in a formal language before they can be studied. Because of this, the basic ways that general mathematical concepts can be manipulated and combined must therefore include those described in the BHK interpretation of constructive logic. Thus this interpretation may be viewed as an extension, rather than a replacement, of the propositions as types view.

This brief survey shows that the interpretation of types and tokens as concepts accords well with the intensional nature of the type theory, and how we think about types, tokens, and their identity.

6 Justifying the Elimination Rule for Identity

Having given an account of the interpretation of tokens and types that is compatible with HoTT's being an autonomous foundation, we now turn to

[35] Note, however, that this is not necessarily the only way One might have non-trivial identifications. One might, for example, simply stipulate that there are two numerically distinct instances of some general concept that are identified (that is, making their distinctness merely a 'brute fact') without providing some other type from which they are abstracted.

the question of how the rules of the theory are to be justified in a way that does not appeal to the homotopy interpretation.

We saw in Section 3.3 that the rules of HoTT^{-} (that is, the fragment of the language without identity types) may be justified via the BHK interpretation of constructive logic. Thus all that remains is the justification of the rules for identity types.

To justify the rules for the type former, recall that the type $\mathtt{Id_C(a, b)}$ corresponds to the proposition that tokens $\mathtt{a : C}$ and $\mathtt{b : C}$ are equal. Forming this type corresponds to expressing the proposition that $\mathtt{a}$ and $\mathtt{b}$ are equal (whereas, recall, proving that they are equal corresponds to constructing a token of that identity type, that is, an identification of $\mathtt{a}$ and $\mathtt{b}$), so it is reasonable that we should be able to form such an identity type for any tokens of any type. It is also reasonable that such an identity type cannot be formed for tokens that are of different types. Since no token can belong to more than one type, tokens of different types must be distinct and it makes no sense to express an identity between them.

The token constructor for identity types allows us to say that any token of any type is identical to itself, that is, that identity is reflexive. It is clear both that this is an essential defining property of identity (so we are justified in having this token constructor), and that we should not, in general, be able to construct identifications between arbitrary pairs of tokens (so no further token constructors are required).[36]

However, the elimination rule for the identity type, namely, the principle of path induction explained in Section 3.4, is not so easily justified. Recall that according to this principle, if a property holds of all trivial self-identifications, then it holds of all identifications. The motivation given in the HoTT Book (and summarized in Section 3.5) depends upon the interpretation of identifications as paths in (homotopy) spaces: since any path is homotopic to a constant path (since it can be continuously retracted), properties of constant paths are shared by all paths. However, as we argue above, such a justification cannot form part of an autonomous foundation for mathematics, for which an alternative account must be found.

In a companion paper (Ladyman and Presnell [2015]), we investigate this problem, and set out a justification for path induction that depends only on HoTT^{-} under our interpretation (which we have argued above is suitable for an autonomous foundation) and two other principles that can be justified independently of the homotopy interpretation and do not depend upon any

[36] One might think that since symmetry and transitivity are essential characteristics of identity, they should also be built into to the definition via new token constructors. However, they don't need to be imposed separately because they can be derived from one of the principles we introduce below, as we show in (Ladyman and Presnell [2015]).

other area of mathematics. Without going into all the technical details, we sketch the argument in the remainder of this section.

One obvious application of identity types is to express facts about the uniqueness of tokens of types. For example, intuitively, there should be exactly one token of the unit type, namely, the token $* : \mathtt{1}$, given by the single token constructor for this type. Normally, in a classical setting, we would express this uniqueness statement by saying that 'there are no tokens of $\mathtt{1}$ other than *', that is, $\neg(\exists \mathtt{x} : \mathtt{1}, * \neq \mathtt{x})$. However, constructively this is equivalent to $\forall \mathtt{x} : \mathtt{1}, \neg\neg(* = \mathtt{x})$, whereas in most applications of uniqueness what we need is the (constructively) stronger statement $\forall \mathtt{x} : \mathtt{1}, (* = \mathtt{x})$. Thus we take the latter form as the statement of uniqueness: that all tokens of the unit type are identical to the token $* : \mathtt{1}$ or, in other words, that there is exactly one token of $\mathtt{1}$ 'up to identity in $\mathtt{1}$'.

More generally, for each type there is a uniqueness principle characterizing the tokens of that type up to suitable identifications. Often the relevant uniqueness principle entails that every token of the type is equal to the output of one of the token constructors for that type—but this is not the case for every type.

In the case of the identity type, the appropriate uniqueness principle cannot make it so that every identification $\mathtt{p} : \mathtt{Id}_{\mathtt{C}}(\mathtt{a}, \mathtt{b})$ is equal to the output of a token constructor, since the constructors only produce identifications of the form $\mathtt{refl}_{\mathtt{x}} : \mathtt{Id}_{\mathtt{C}}(\mathtt{x}, \mathtt{x})$, and we cannot identify tokens of different types.

To characterize the uniqueness principle for identity types, we therefore need to define a type whose tokens are representatives or counterparts of arbitrary identifications, such as $\mathtt{p} : \mathtt{Id}_{\mathtt{C}}(\mathtt{a}, \mathtt{b})$, and trivial self-identifications, such as $\mathtt{refl}_{\mathtt{x}} : \mathtt{Id}_{\mathtt{C}}(\mathtt{x}, \mathtt{x})$. In this type, we will be able to express a uniqueness principle that states that the counterpart of any arbitrary $\mathtt{p} : \mathtt{Id}_{\mathtt{C}}(\mathtt{a}, \mathtt{b})$ is identical to that of some trivial self-identification.

For this purpose we define, for each $\mathtt{a} : \mathtt{C}$, the 'based identity type' at $\mathtt{a}$

$$\mathtt{E}(\mathtt{a}) :\equiv \sum_{\mathtt{z}:\mathtt{C}} \mathtt{Id}_{\mathtt{C}}(\mathtt{a}, \mathtt{z}).$$

The tokens of this type are pairs $(\mathtt{b}, \mathtt{p})$ consisting of token $\mathtt{b} : \mathtt{C}$ and identification $\mathtt{p}$ between that token and the given token, $\mathtt{a} : \mathtt{C}$. In particular, one token of this type is $(\mathtt{a}, \mathtt{refl}_{\mathtt{a}})$. Thus for any identification $\mathtt{q} : \mathtt{Id}_{\mathtt{C}}(\mathtt{x}, \mathtt{y})$, we can find a counterpart, $(\mathtt{y}, \mathtt{q})$, to it in the based identity type, $\mathtt{E}(\mathtt{x})$, alongside a counterpart to the trivial self-identification, $\mathtt{refl}_{\mathtt{x}}$.[37]

[37] We could instead define the free identity type whose tokens are triples $(\mathtt{a}, \mathtt{b}, \mathtt{p})$ consisting of two tokens of C and an identification between them. This type also contains a counterpart to every identification, alongside counterparts to trivial self-identifications. However, the justification of the elimination rule for identity types is more easily expressed in terms of the based identity type defined above, so we restrict attention to this. We thus directly justify what is known as 'based path induction', from which path induction also follows, as shown in (HoTT Book [2013], Section 1.12.2).

In standard approaches to mathematics, we would expect that the only token of $\mathtt{C}$ that is identical with $\mathtt{a}$ is $\mathtt{a}$ itself, and moreover that it should be self-identical in just one way. Thus we would expect $\mathtt{E}(\mathtt{a})$ to have only a single token, just as we expect the unit type to have just a single token. As with the unit type, we can express this by saying that 'all tokens of $\mathtt{E}(\mathtt{a})$ are identical to $(\mathtt{a}, \mathtt{refl}_{\mathtt{a}})$', that is, that $\mathtt{E}(\mathtt{a})$ has exactly one token 'up to identity in $\mathtt{E}(\mathtt{a})$'. Formally, we say that for any token $(\mathtt{b}, \mathtt{p}) : \mathtt{E}(\mathtt{a})$, there is an identification between it and $(\mathtt{a}, \mathtt{refl}_{\mathtt{a}})$, which we express by saying that the following type

$$\prod_{(\mathtt{b},\mathtt{p}) : \mathtt{E}(\mathtt{a})} \mathtt{Id}_{\mathtt{E}(\mathtt{a})}((\mathtt{a}, \mathtt{refl}_{\mathtt{a}}), (\mathtt{b}, \mathtt{p}))$$

is inhabited. This is the uniqueness principle for identity types.

Note that even with the addition of this principle, it does not follow that all identifications are trivial self-identifications (which, as discussed above, cannot even be expressed in the language); nor does it follow that all self-identifications are trivial. Thus the way identity is treated in HoTT differs from standard approaches in mathematics and logic, but the uniqueness principle for identity types goes some way towards reconciling them.

The second principle we introduce is *substitution salva veritate*: if tokens $\mathtt{a}$ and $\mathtt{b}$ are identical, then anything that is true of one must be true of the other. This is an essential defining property of identity.[38] Combining these two principles, we derive the elimination rule for identity types: if we can prove that some property holds of all trivial self-identifications—or, more precisely, all pairs $(\mathtt{a}, \mathtt{refl}_{\mathtt{a}})$ for all $\mathtt{a} : \mathtt{C}$—then it must likewise hold for any pair $(\mathtt{b}, \mathtt{p}) : \mathtt{E}(\mathtt{a})$, since the uniqueness principle states that this is identical to $(\mathtt{a}, \mathtt{refl}_{\mathtt{a}})$.

Thus by applying the uniqueness principle for identity types and the principle of *substitution salva veritate*, each of which is justified on purely pre-mathematical grounds, we derive the elimination rule for identity types. Formally, this rule, called 'based path induction', states that for any type $\mathtt{C}$, any token $\mathtt{a} : \mathtt{C}$, and any predicate $\mathtt{K}$ on $\mathtt{E}(\mathtt{a})$, there is a function of type

$$\mathtt{K}(\mathtt{a}, \mathtt{refl}_{\mathtt{a}}) \rightarrow \prod_{(\mathtt{b},\mathtt{p}) : \mathtt{E}(\mathtt{a})} \mathtt{K}(\mathtt{b}, \mathtt{p}).$$

This concludes the justification of the treatment of identity in HoTT in a manner that is compatible with its being an autonomous foundation for mathematics.[39]

[38] One might object that this principle is in conflict with an intensional treatment of types. In Section 8.5 we explain why this does not pose a problem.

[39] For a detailed version of this argument, along with proofs that other properties of identity, such as symmetry and transitivity follow from *substitution salva veritate*, see (Ladyman and Presnell [2015]).

7 The Foundations of HoTT without Homotopy

This section gives answers to the questions of Section 2 for the new interpretation of HoTT presented in this article.

7.1 Framework

The theory makes use of the language of HoTT, as sketched in Section 3. Complex expressions are constructed systematically and precisely from basic ones by application of the rules. HoTT is not formulated using a separately defined formal logical language like first-order predicate calculus in the way that axiomatic ZFC is, but rather incorporates constructive logic directly into the mathematical theory, and propositional and predicate logic have analogues within it. The role of logical axioms is played by the rules of type formation (for example, the rule that states that for any types $\mathtt{A}$ and $\mathtt{B}$, there is a product type $\mathtt{A} \times \mathtt{B}$).

Other axioms and premises can be adopted by introducing tokens of types expressing the corresponding propositions. For example, while LEM does not hold as a law of the constructive logic incorporated into HoTT, for any given type, $\mathtt{P}$, we may assume as a premise of an argument that $\mathtt{P}$ is 'decidable' by positing a token of type $\mathtt{P} + (\mathtt{P} \to \mathtt{0})$.[40]

7.2 Semantics

Types and tokens are interpreted as concepts (rather than spaces, as in the homotopy interpretation). In particular, a type is interpreted as a general mathematical concept (for example, 'natural number'), while a token of a given type is interpreted as a more specific mathematical concept *qua* instance of the general concept (for example, 2 *qua* natural number). This accords with the fact that each token belongs to exactly one type. Since 'concept' is a pre-mathematical notion, this interpretation is admissible as part of an autonomous foundation for mathematics. Note that this is compatible with the popositions-as-types interpretation of Section 3.3, as explained at the end of Section 5.

The theory is intensional: types are distinguished from one another by their definitions (that is, by their constructions), rather than by their extensions. Of course, since each token belongs to exactly one type, there is no sense in which two inhabited types could have the same extension. But uninhabited types—

[40] Another important example is the univalence axiom (HoTT Book [2013], Section 2.10), which is central to the research programme developing HoTT. While univalence cannot be derived as a theorem from the basic framework of HoTT, many proofs use univalence by assuming the existence of a token of an appropriate type. As noted in Footnote 6, in this article we consider only the basic language of HoTT, not including univalence.

for example, 'even divisor of 9' and 'even divisor of 11'—are treated as distinct. Moreover, any two types that might be understood as containing (representatives of) the same mathematical entities under different presentations—for example, 'positive natural number less than 3' and 'natural number n for which the Fermat equation $x^n + y^n = z^n$ has at least one solution in the positive integers'—are nonetheless considered to be distinct types.[41] Thus any distinction that can be made gives rise to a distinction between types. The justification given in the HoTT Book for using a theory that is intensional in this way is the fact that it enables the homotopy interpretation (HoTT Book [2013], Chapter 1 Notes, Chapter 2 Notes).

Expressions in the language are the names of types and tokens. Those naming types correspond to propositions. A proposition is 'true' just if the corresponding type is inhabited (that is, there is a token of that type, which we call a 'certificate' to the proposition). For example, any isomorphism between two structures is a certificate to the proposition that the structures are isomorphic. The negation of a proposition, P, is represented by the type $\mathtt{P} \to \mathtt{0}$, where $\mathtt{P}$ is the type corresponding to proposition P and $\mathtt{0}$ is a type that by definition has no token constructors (corresponding to a contradiction). The logic of HoTT is not bivalent, since the inability to construct a token of $\mathtt{P}$ does not guarantee that a token of $\mathtt{P} \to \mathtt{0}$ can be constructed, and vice versa.

The rules governing the formation of types are understood as ways of composing concepts to form more complex concepts, or as ways of combining propositions to form more complex propositions. They follow from the Curry–Howard correspondence between logical operations and operations on types. However, we depart slightly from the standard presentation of the Curry–Howard correspondence and, in particular, from the way that Martin-Löf ([1996]) presents it, in that the tokens of types are not to be thought of as 'proofs' of the corresponding propositions, but rather as certificates to their truth. A proof of a proposition is the construction of a certificate to that proposition by a sequence of applications of the rules. Two different such processes can result in the construction of the same token, and so proofs and tokens are not in one-to-one correspondence.

The rules of the fragment of HoTT that we called HoTT$^-$ are justified as the instantiations of the elementary logical operations such as conjunction and disjunction, all of which can be understood and motivated pre-mathematically. The specific form of the rules comes from the interpretation of logical reasoning as the manipulation of certificates to

[41] The addition of the univalence axiom (HoTT Book [2013], Section 2.10) weakens these distinctions, since it gives identifications between types that are equivalent in a suitable sense. But as noted above, the theory considered here does not include univalence.

propositions, which leads (via the BHK interpretation) to the constructive logic that is used in HoTT.[42]

The type former and token constructor for the identity type are easily justified, as outlined in Section 6. In Section 6 we sketch how to derive path induction, the elimination rule for identity types, from pre-mathematical notions. (A more detailed discussion of this is given in our companion paper; see Ladyman and Presnell [2015].)

When we work formally in HoTT, we construct expressions in the language according to the formal rules. These expressions are taken to be the names of tokens and types of the theory. The rules are chosen such that if a construction process begins with non-contradictory expressions that all name tokens (that is, none of the expressions are empty names), then the result will also name a token (that is, the rules preserve non-emptiness of names). Hence, our interpretation, according to which expressions name tokens and types understood as concepts, connects the intensionality of HoTT with its constructive nature. This is another significant departure from Martin-Löf's ideas.[43]

7.3 Metaphysics

In the standard presentation of HoTT, nothing is said about the metaphysics. In particular, it is not clear whether the tokens of types literally are mathematical objects (for example, natural numbers) or whether they represent them in some sense. We are free to apply any metaphysical interpretation we prefer. Nothing is said about applicability in the physical sciences.

Since we interpret tokens and types as concepts, the only metaphysical commitment required is to the existence of concepts themselves. That human thought involves concepts is an uncontroversial position, and our interpretation does not require that concepts have any greater metaphysical status than is commonly attributed to them (Isaacson [1994]).

Just as the existence of a concept such as 'unicorn' does not require the existence of actual unicorns, likewise our interpretation of tokens and types as mathematical concepts does not require the existence of mathematical objects. However, it is compatible with such beliefs. Thus, for example, a Platonist can take the concept 'equilateral triangle' to be the concept corresponding to the abstract equilateral triangle (after filling in some account of how we come to

[42] We are not here committed to inferentialism about the meaning of the logical constants. However, in so far as they are represented in HoTT, which derives from the work of Martin-Löf ([1996]), inferentialism is presupposed. We are grateful to an anonymous referee for pressing us on this.

[43] Martin-Löf ([1995], [1998]) motivates the constructive nature of his type theory on the basis of a verificationist theory of meaning and a proof-theoretic semantics, according to which the meaning of a proposition is given by an account of what would count as a proof of it, and the meaning of the logical constants is given by the relevant token constructors and type formers. Martin-Löf's ([1996]) philosophy also gives a more prominent role to the notion of judgements.

know about these abstract objects in a way that lets us form the corresponding concepts).

Even without invoking mathematical objects to be the targets of mathematical concepts, one could still maintain that concepts have a mind-independent status. However, this is not a necessary part of the interpretation, and one could instead take concepts to be mind-dependent, with corresponding implications for the status of mathematics itself.

7.4 Epistemology

In the interpretation proposed here, as in the standard presentation of HoTT, since there are no metaphysical commitments to mathematical objects, there is no issue of explaining how we come to know about abstract objects, The corresponding problem of explaining how we have epistemic access to the concepts we form is one that is not unique to mathematics. In any case, almost all accounts of mathematics must give some account of mathematical concepts and mathematical thought, even if it is a reductionist one.

Since complex concepts are formed step-by-step from more basic ones, our access to any given concept can be traced backward via definitions to whatever elementary concepts are assumed to be given. The truth of a proposition is demonstrated by exhibiting a certificate, and a proof is a step-by-step construction of a certificate to the conclusion from certificates to the premises. Any claim to mathematical knowledge that can be demonstrated by means of a proof in constructive logic gives a specific way of constructing a certificate to the corresponding proposition. This is true even of proofs of negated propositions, since the classical technique of indirect proof (proving P by deriving contradiction from the negation of P) is disallowed. HoTT uses constructive logic, so proofs always result in positive evidence for their conclusions. Moreover, HoTT can accommodate proofs that rely upon non-constructive methods—for example, those using instances of LEM—since these principles can be explicitly added as additional premises in a proof.

The sketch of an account of mathematical knowledge given here is, of course, not the last word on the matter, and we are not committed to the claim that only propositions that can be proved within the system should count as mathematical knowledge. For example, if one had reason to believe that there were non-deductive routes to mathematical knowledge (as suggested by Paseau [2015]), then things taken to be known by such means may be posited as axioms or assumptions. So while the existence of a proof in the system is sufficient for mathematical knowledge, it is not claimed to be necessary, and we may allow that one can have reason to believe in the existence of tokens of types without being able to construct those tokens.[44]

Where a proof depends upon premises it provides only conditional knowledge of its conclusion. In this case, the assumed premises—and, moreover, the particular certificates to those premises that are assumed—are made completely explicit.

As detailed above, the constructive logic guarantees that if the input expressions name tokens (that is, they are not empty names), then the output expressions name tokens as well. We don't know, in general, that arbitrary types are inhabited. There are no other axioms of the basic theory.[45] The justification of axioms introduced to recover extant mathematics—such as, for example, the natural numbers—depends only upon our ability to form the relevant concepts and not on our epistemic access to mathematical objects.

Note that the incorporation of constructive logic into HoTT does not derive from nor necessitate any strong position on the nature of mathematical truth itself. In particular, it does not require that we equate mathematical truth with provability, nor does it require that we deny that unproved (or unprovable) mathematical statements have truth values. While some approaches to mathematics (such as that of Brouwer [1981]) begin from a view about the nature of mathematical truth and conclude from this that reasoning should be constructive, it is entirely possible to take a much more modest or neutral position regarding mathematical truth, and adopt constructive reasoning for pragmatic or methodological reasons.

7.5 Methodology

Mathematical reasoning in HoTT is broadly similar to any other foundational system, in as much as it proceeds by definition, formulation of propositions, and proofs. We begin by formulating types corresponding to the kinds of mathematical entities under discussion. Then from these we form types expressing the proposition to be proved and the premises to be assumed. We introduce names for the tokens of the premises we assume given, and then seek to construct a token of the conclusion.

While many presentations of HoTT emphasize its use as a formal language for mathematics, as with any mathematical formalism, the language of HoTT may also be used in an informal or semi-formal manner. Just as set-theoretic reasoning can be used in arguments that are not rendered into completely formal manipulations in first-order logic, likewise reasoning in HoTT may

[44] Hence this account does not fall foul of Gödel's incompleteness theorem. We are grateful to an anonymous referee for pressing us on these issues.

[45] As noted above, the univalence axiom is often assumed in work in HoTT, but we take this to be outside the core of HoTT and thus beyond the scope of the present article.

intermix mathematical notation with human-language descriptions, omitting details for the reader to fill in, as appropriate to the context. Indeed, such a style is used in much of the HoTT Book.

Note that this methodology does not require that every type involved be completely formalized down to the smallest detail. We can just as well introduce placeholder types, so long as whatever particular properties we require of them for a given proof can be derived or are explicitly assumed. In particular, we can still reason about entities whose existence cannot be constructively derived (such as non-principal ultrafilters) by assuming their existence (and any relevant properties) as premises.

An important difference between HoTT and other foundations, such as ZFC set theory, is the relative ease with which informal use of HoTT can be fully formalized in a computer programming language that can be used directly to construct completely formal proofs to check informal reasoning. In mathematics based in ZFC set theory, formal proofs can in principle be rendered completely formal but in HoTT, the practicality of computer-assisted reasoning and proof verification narrows the gap between everyday practice and idealized formal reasoning. This is a major motivation for HoTT. Arguably, computability requires, or at least is facilitated by, a constructive framework. If we wish to ensure that we only assert mathematical claims for which we have direct and positive evidence, then a constructive methodology guarantees this.

8 Possible Objections to this Account

In this section we address a number of objections that may be raised to the interpretation set out above.

8.1 A constructive foundation for mathematics?

As noted in Section 2, some mathematicians and philosophers may object to the claim that HoTT could be a foundation for mathematics on the grounds that it cannot even serve as an adequate framework, since its logic is constructive rather than classical. On some views of what is required of a framework for mathematics, any adequate framework must enable us to derive all the theorems of classical mathematics just as they are standardly stated, without the need to modify them. On this view, no constructive foundation could be adequate, since many theorems of standard mathematics, in their usual formulation, require classical principles for their proofs.[46]

[46] Consider, for example, Hilbert's famous objection that 'taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists' (Hilbert [1976], p. 476).

For example, one version of the intermediate value theorem of analysis states that for any continuous function, $f : \mathbb{R}\to\mathbb{R}$, and for any values $a < b$ such that $f(a) < 0$ and $f(b) > 0$, there is a real number $a < x < b$ such that $f(x) = 0$. This theorem is not provable constructively (roughly, because any constructive proof must provide a means of finding the required x to arbitrary precision). However, there are several variants of the theorem that are constructively valid. For example, for any continuous function, f, with $f(a) < 0$ and $f(b) > 0$, we can prove that for any $\epsilon > 0$, there is a real number $a < x < b$ such that $|f(x)| < \epsilon$. To judge whether a constructive foundation is adequate, one must decide, amongst other things, whether the ability to recover such weakenings of classical theorems is sufficient.

One could take the position that we should not assume in advance that the foundations of mathematics should be classical or, more generally, that it is not *a priori* obvious what fundamental principles should be adopted in a foundation for mathematics. For example, there is no principled reason why the continuum hypothesis should not be appended to the axioms of ZFC to form an alternative foundation for mathematics; but we should not then judge ZFC itself as inadequate for being unable to recover all the theorems provable in ZFC + CH. Likewise then, it is no more reasonable to judge a constructive foundation inadequate merely because it fails precisely to recover all theorems provable in ZFC. The defender of a constructive foundation could argue, then, that each candidate foundation should be judged on its own merits, rather than being compared against ZFC as a 'default' or 'standard' foundation.

Finally, as noted in Section 7.4, if non-constructive principles, such as particular instances of LEM, are required for a proof then they may be added as assumptions. Thus while such principles are not provided as part of the framework, they are compatible with it and may be adopted as required.

Another important difference between the framework of HoTT and that of ZFC set theory is the treatment of functions. Whereas in ZFC set theory functions are defined as relations satisfying certain conditions and are therefore subsets of the product of their domain and codomain, in HoTT functions are defined via a process of substitution and are therefore more like procedures or algorithms in computer science. Thus not every 'function' of ZFC set theory counts as a function in this system. This is an essential feature of the constructive logic used in HoTT and so objections to this approach to functions are again objections to the use of this constructive logic.

We will not pursue these issues further, simply noting (as we did in Section 2) that if one does not accept that a constructive framework can be adequate, then the question of whether HoTT can be a foundation for mathematics is immediately answered in the negative.[47]

8.2 What are concepts?

A foundation for mathematics is supposed to clear up the mystery of what mathematical entities are and how we come to know about them, given that (according to many interpretations) they are abstract entities with no causal powers. Since we interpret the language of this foundation for mathematics in terms of concepts, it may be objected that we have simply traded one mystery for another, namely, the question of what concepts are and what their metaphysical status is.

However, although we do invoke concepts in our interpretation of tokens and types, the features of concepts that we rely upon in Section 5.2 are only those straightforward features that follow from our intuitive understanding of concepts. Thus we do not depend upon any advanced or intricate theory of concepts, and therefore do not need to give a comprehensive detailed account of concepts. Furthermore, concepts are arguably needed for any account of thought more generally and are not an additional requirement for mathematics in particular.

Since concepts play such a central role in this proposed account of HoTT as a foundation for mathematics, some features of the foundation provided by HoTT will depend upon how we think about concepts. For example, as mentioned in Section 7.3, the mind-dependent or mind-independent status of mathematics will depend, on this interpretation, on whether concepts themselves are mind-dependent or mind-independent. Similarly, one might be concerned to explain how different mathematicians appear to be talking about the same domain when (on some accounts) each has his or her own independent concepts.

We acknowledge this dependence but do not take it to be an objection to the account of HoTT as a foundation. Rather, it means that there are a number of interpretational varieties of the foundation available, which append different accounts of concepts and thus give different views on the status of mathematical entities. Moreover, interpreting mathematical talk as talk about concepts integrates problems in philosophy of mathematics into this wider domain. So, for example, questions about the inter-subjectivity of mathematics are subsumed into questions about the inter-subjectivity of concepts more generally.[48]

[47] For further defence of the adequacy of constructive mathematics, see, for example, (McCarty [2005]; Richman [1996]).

[48] Of course, the answers we give in the case of mathematics might still be specific to that domain. For example, even without giving a full account of how concepts can be shared between subjects, we might still be able to give an account of how the concepts of mathematics, being rigorously defined in a formal language, can be shared.

8.3 Isn't this just Brouwerian intuitionism?

According to the interpretation presented here, mathematical language refers to concepts rather than mathematical objects, and that mathematical reasoning is the manipulation of concepts using constructive logic. In these respects, it resembles Brouwerian intuitionism (Brouwer [1981]), but there are a number of ways in which the two positions differ.

Brouwer's intuitionism positively rejected consideration of mathematical objects, whereas our account is entirely compatible with a Platonist position (since the existence of objects that can be treated by the mathematician entails the existence of concepts under which they can be understood).

Brouwer's position puts the activity of the human mind at the centre, whereas our interpretation is compatible with an understanding of concepts that grants them a mind-independent status. In particular, our interpretation is compatible with mathematical reasoning being carried out by an automated process in a computer, without any human being involved (after the initial setup of the computer program itself, of course). So, for example, a sufficiently large and complex calculation in a computer might make use of types that correspond to concepts that have never been conceived by a human reasoner, and may (for all we know) be too complex for any human mind to conceive.[49]

Brouwer held that 'intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time [which] may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory' (Brouwer [1981], pp. 4–5). No such hypothesis plays any role in our interpretation of HoTT as a foundation for mathematics, and indeed we give no account of how mathematical concepts originate. Moreover, while we leave it open that one could take concepts to be necessarily prelinguistic, any concepts that are involved in mathematical activity must, on our account, be given a precise and rigorous definition in the formal language of HoTT.

Finally, Brouwer's intuitionism is often interpreted as taking the truth and falsity of propositions to be a temporal matter, with propositions only gaining a truth value when they are first proved or disproved and having no truth value at all before then. Our interpretation makes no such claim, and allows that propositions may have truth values—or rather, that types may have tokens—that are simply unknown to us.

[49] Although this last point would not be relevant if we took the reasoner in intuitionism to be the idealized 'creating subject' (Brouwer [1948]).

8.4 Duplicated objects

Mathematicians standardly consider, for example, the number 2 as a single entity that appears in multiple different domains—the natural numbers, the reals, the complex numbers, and in subsets such as the even numbers and the prime numbers. However, in the formalism of HoTT, this cannot be the case, since each token belongs to exactly one type. This appears to require us needlessly to multiply entities, producing ‘clones’ or ‘counterparts’ that inhabit different types. This, it might be argued, is inelegant and in conflict with our intuitions.

In response, we recall that the tokens of the theory are interpreted not as mathematical objects, but as specific concepts *qua* instances of general concepts. Even if we think that ‘the number 2’ is a singular entity, it is clear that the conception of 2 as a rational number is distinct from the conception of 2 as a natural number. This is a natural distinction, not one artificially imposed. Moreover, we have no concept of ‘the number 2’ except as a member of some number system—we do not conceive of the number 2, separately conceive of the natural numbers, and then determine that the former is a member of the latter.

Rather, it is these concepts—‘2 *qua* natural number’, ‘2 *qua* rational number’, and so on—to which we can plausibly claim to have direct access. We have no concept of ‘the number 2 *simpliciter*’, and any putative access to ‘the number 2 itself’ (as a mathematical object) is notoriously shrouded in mystery.

8.5 Intensionality and *substitution salva veritate*

Our justification of path induction (the elimination rule for identity types) given in Section 6 depends upon the principle of *substitution salva veritate*. However, intensionality generates opaque contexts, namely, ones in which co-referring expressions cannot be substituted while preserving the truth of a sentence. Since the treatment of types in HoTT is intensional, surely *substitution salva veritate* is routinely violated and so cannot be a defining principle of identity?

For example, in sentences describing the contents of a person’s thoughts or beliefs, if that person is unaware of the fact that two names or descriptions refer to the same entity, then substituting one for the other may change the truth of a sentence. For example, the sentence ‘*S* is thinking about the inventor of bifocal glasses’ can be true while the sentence ‘*S* is thinking about the first US Postmaster General’ is arguably false, if *S* does not know that both of these descriptions pick out Benjamin Franklin.

The standard diagnosis of this phenomenon is that in some contexts names and descriptions are implicitly associated with a 'mode of presentation'. When a sentence omits this mode of presentation (as is usually the case) it can be rendered ambiguous, and filling in different modes of presentation can change the truth value. Thus, in the case above, *S* is thinking about Benjamin Franklin under the mode of presentation 'inventor of bifocal glasses' and not under the mode of presentation 'first US Postmaster General'. Thus *S*'s thinking about Franklin can render the first sentence true and the second sentence false, even though the object of *S*'s thought, Benjamin Franklin, is the extension of both descriptions.

Arguably, eliminating ambiguity about intensions solves this problem, and this is the case according to our interpretation of HoTT. The intension of a token is the type that it belongs to—for example, the intension of '2 *qua* rational number' is 'rational number'. Since each token belongs to exactly one type, there can be no ambiguity about the intension under which it is considered, so the intensional treatment of types does not generate an opaque context, and *substitution salva veritate* is not violated.

9 Conclusion

In this article we granted that HoTT is an adequate framework for mathematical practice and addressed the question as to whether, in addition, it is a candidate for being an autonomous foundation for mathematics, in the strong sense that involves providing an account of the semantics, metaphysics, and epistemology. Section 4 argued that under the standard interpretation, HoTT is not autonomous. In Section 5 we introduced the types-as-concepts interpretation and argued that, so construed, HoTT is such a candidate foundation for mathematics. While the framework in question is that presented in the HoTT Book ([2013], Chapter 1) (without function extensionality and univalence; see Footnote 6), this article presents a new interpretation of the tokens and types of the theory, and an account of the metaphysical commitments and the epistemology that goes with it. Whether this interpretation can be extended to cover function extensionality, univalence, and other additions to the theory is a matter for further research.

The types-as-concepts interpretation of the theory presented here is metaphysically relatively conservative. No special ontology of abstract objects is required for mathematics. If concepts are regarded as abstract objects, then either we need them anyway to account for non-mathematical thought, or some reductive or eliminative account of them can be given, in which case it can be applied to mathematical concepts as well. We have no such account in mind, but do not rule one out, and we do not address the ontology of concepts further here.

As far as epistemology goes, the present view requires that concepts can be accessed by thought, and again there is no special mystery about this in the mathematical case. When it comes to knowledge of mathematical truths, where these are construed as conceptual facts, any philosophy of mathematics that respects mathematical practice must take proof to be the primary means by which mathematical knowledge is established. Any informal proof in HoTT can always be translated into a completely formal proof and the logic of the latter is constructive. Thus knowledge of the truth of theorems is reduced to knowledge of the existence of certificates to those theorems, and a proof is a complete and explicit specification of a method to produce such a certificate by the application of simple rules.

Note that none of this rules out an ontology of abstract objects. However, we do not need to posit the latter in order to do mathematics. Our interpretation is compatible with stronger positions that have such an ontology of mathematics, but it does not give any additional support to them. Similarly, the epistemological position proposed here provides a basis for mathematical knowledge via constructive proofs, but does not rule out that it may be had in other ways, such as via proofs in classical logic (or even by some non-deductive means, Paseau [2015]). For example, a particular proof may assume the applicability of LEM or some other additional premise that is not a consequence of the axioms of the theory; such a proof is then conditional on these assumptions, which must be made explicit. Unlike some strong Intuitionist positions, the epistemology proposed here does not rule out the use of LEM, it simply requires that such uses be explicitly flagged up as assumptions that must either be justified or assumed as premises of the proof. In short, the metaphysical and epistemological positions proposed here do not restrict the ontological or mathematical assertions that can be made, but they do limit the assertions for which we can claim justification without further premises.

9.1 Advantages of this foundation

HoTT understood as above or as in the HoTT Book has many advantages as a foundation for mathematics. Type theories help us avoid paradox and errors, and the constructive framework of HoTT facilitates automatic proof verification. The intensionality of the theory allows for the finest of distinctions to be made. In terms of flexibility and the naturalness with which structures can be represented, it is like category theory rather than set theory; but HoTT also unifies logic with mathematics via the Curry–Howard correspondence. As

Awodey ([2014]) argues, HoTT is also a promising framework for structuralism in the philosophy of mathematics.

In mathematical practice, every well-formulated definition is the careful expression of some concept, usually formed by the logical combination of simpler concepts. We have direct access to concepts, as well as to pictures and symbols, in thinking about mathematics. Mathematical concepts are often taken to be about mathematical objects, but (even setting aside problems of the nature of these abstract objects and our access to them) the relationship between concepts and objects is not straightforward. It can be a matter of great mathematical effort to determine that a given mathematical concept denotes a particular object, and it is often a substantial mathematical achievement to show that two concepts are extensionally equal.[50]

Hence, it is plausible that concepts and the definitions that directly correspond to them should be taken as the primary entities of mathematics, while the objects that they (appear to) denote should be taken as at best secondary and at worst redundant. It follows that our foundational language for mathematics should be understood as picking out concepts rather than objects. This implies that the language should be intensional, maintaining a distinction between conceptually distinct descriptions even when they have the same extension. This is in contrast to the extensional foundations given by ZFC set theory, for example, which takes the objects (that is, sets) as primary and therefore collapses together co-extensional definitions.

Our types-as-concepts interpretation of HoTT motivates the intensionality of the theory, coheres with its constructive nature, explains how identity types are thought of and used, and is independent of, but compatible with, the homotopy interpretation. Whether or not these advantages are sufficient to show that HoTT so construed or otherwise should be adopted as a foundation for mathematics is a matter for future work.

Acknowledgements

We are very grateful to four anonymous referees for the journal and to Steve Awodey, Øystein Linnebo, Richard Pettigrew, Urs Schreiber, Teru Thomas, Patrick Walsh, and John Wigglesworth for reading and commenting on earlier drafts of this article. We are also grateful to participants and organizers of the Second Symposium on the Foundations of Mathematics, Institute of Philosophy, London, January 2015, at which some of these ideas were presented.

[50] Consider, for example, the concepts 'positive natural number less than 3' and 'natural number n for which the Fermat equation $x^n + y^n = z^n$ has at least one solution in the positive integers', noted in Section 7.2.

The research on which this article is based, ‘Applying Homotopy Type Theory in Logic, Metaphysics, and Philosophy of Physics’, is funded by a Leverhulme Trust Research Project Grant (RPG-2013-228).

James Ladyman
Department of Philosophy
University of Bristol
Bristol, UK
james.ladyman@bristol.ac.uk

Stuart Presnell
Department of Philosophy
University of Bristol
Bristol, UK
stuart.presnell@bristol.ac.uk

References

Aharonov, Y. and Bohm, D. [1959]: ‘Significance of Electromagnetic Potentials in the Quantum Theory’, *Physical Review*, **115**, pp. 485–91.

Awodey, S. [1996]: ‘Structure in Mathematics and Logic: A Categorical Perspective’, *Philosophia Mathematica*, **4**, pp. 209–37.

Awodey, S. [2014]: ‘Structuralism, Invariance, and Univalence’, *Philosophia Mathematica*, **22**, pp. 1–11.

Awodey, S and Warren, M. [2009]: ‘Homotopy Theoretic Models of Identity Types’, *Mathematical Proceedings of the Cambridge Philosophical Society*, **146**, pp. 45–55.

Bell, J. L. [2008]: *A Primer of Infinitesimal Analysis*, Cambridge: Cambridge University Press.

Benacerraf, P. and Putnam, H. (*eds*) [1983]: *Philosophy of Mathematics: Selected Readings*, Cambridge: Cambridge University Press.

Brouwer, L. E. J. [1948]: ‘Essentially Negative Properties’, *Indagationes Mathematicae*, **10**, pp. 322–3.

Brouwer, L. E. J. [1981]: *Brouwer's Cambridge Lectures on Intuitionism*, Cambridge: Cambridge University Press.

Church, A. [1936]: ‘An Unsolvable Problem of Elementary Number Theory’, *American Journal of Mathematics*, **58**, pp. 345–63.

Colyvan, M. [2001]: *The Indispensability of Mathematics*, New York: Oxford University Press.

Curry, H. [1934]: ‘Functionality in Combinatory Logic’, *Proceedings of the National Academy of Sciences*, **20**, pp. 584–90.

Detlefsen, M. [2005]: ‘Formalism’, in S. Shapiro (*ed.*), *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford: Oxford University Press, pp. 236–317.

Field, H. H. [1980]: *Science without Numbers: A Defence of Nominalism*, Princeton: Princeton University Press.

Healey, H. [1997]: 'Nonlocality and the Aharonov–Bohm Effect', *Philosophy of Science*, **64**, pp. 18–41.

Hilbert, D. [1976]: 'The Foundations of Mathematics', in J. van Heijenoort (*ed.*), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, Cambridge, MA: Harvard University Press, pp. 464–79.

Howard, W. A. [1980]: 'The Formulae as Types Notion of Construction', in J. P. Seldin and J. R. Hindley (*eds*), *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, Boston, MA: Academic Press, pp. 479–90.

Isaacson, D. [1994]: 'Mathematical Intuition and Objectivity', in A. George (*ed.*), *Mathematics and Mind*, New York: Oxford University Press, pp. 118–40.

Ladyman, J. and Presnell, S. [2014]: 'A Primer on Homotopy Type Theory', available at <http://www.bristol.ac.uk/homotopy-type-theory/>.

Ladyman, J and Presnell, S. [2015]: 'Identity in Homotopy Type Theory, Part I: The Justification of Path Induction', *Philosophia Mathematica*, **23**, pp. 386–406.

Linnebo, Ø. and Pettigrew, R. [2011]: 'Category Theory as an Autonomous Foundation', *Philosophia Mathematica*, **19**, pp. 227–54.

Martin-Löf, P. [1974]: 'An Intuitionistic Theory of Types: Predicative Part', in H. E. Rose and J. C. Shepherdson (*eds*), *Logic Colloquium 73*, Amsterdam: North-Holland, pp. 73–118.

Martin-Löf, P. [1995]: 'Verificationism Then and Now', in W. DePauli-Schimanovich, E. Köhler and F. Stadler (*eds*), *The Foundational Debate: Complexity and Constructivity in Mathematics and Physics*, Dordrecht: Kluwer, pp. 187–96.

Martin-Löf, P. [1996]: 'On the Meanings of the Logical Constants and the Justifications of the Logical Laws', *Nordic Journal of Philosophical Logic*, **1**, pp. 11–60.

Martin-Löf, P. [1998]: 'Truth and Knowability: On the Principles of C and K of Michael Dummett', in H. G. Dales and G. Oliveri (*eds*), *Truth in Mathematics*, Oxford: Clarendon Press, pp. 105–14.

Mayberry, J. P. [1994]: 'What Is Required of a Foundation for Mathematics?', *Philosophia Mathematica*, **2**, pp. 16–35.

McCarty, D. C. [2005]: 'Intuitionism in Mathematics', in S. Shapiro (*ed.*), *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford: Oxford University Press, pp. 356–86.

Paseau, A. [2015]: 'Knowledge of Mathematics without Proof', *British Journal for the Philosophy of Science*, **66**, pp. 775–9.

Putnam, H. [1967]: 'Mathematics without Foundations', *Journal of Philosophy*, **64**, pp. 5–22.

Richman, F. [1996]: 'Interview with a Constructive Mathematician', *Modern Logic*, **6**, pp. 247–71.

Shapiro, S. [1991]: *Foundations without Foundationalism: A Case for Second-order Logic*, Oxford: Oxford University Press.

Shapiro, S. (*ed.*) [2005]: *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford: Oxford University Press.

The Univalent Foundations Program [2013]: *Homotopy Type Theory: Univalent Foundations of Mathematics*, Princeton: Institute for Advanced Study, available at <homotopytypetheory.org/book>.

Troelstra, A. S. [2011]: 'History of Constructivism in the 20th Century', in J. Kennedy, and R. Kossak (*eds*), *Set Theory, Arithmetic, and Foundations of Mathematics*, Cambridge: Cambridge University Press, pp. 150–79.