

Complex Systems from the Perspective of Category Theory: II. Covering Systems and Sheaves

Abstract

Using the concept of adjunction, for the comprehension of the structure of a complex system, developed in Part I, we introduce the notion of covering systems consisting of partially or locally defined adequately understood objects. This notion incorporates the necessary and sufficient conditions for a sheaf theoretical representation of the informational content included in the structure of a complex system in terms of localization systems. Furthermore, it accommodates a formulation of an invariance property of information communication concerning the analysis of a complex system.

Keywords : Complex Systems, Information Structures, Localization Systems, Coverings, Adjunction, Sheaves.

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1 Recapitulation of the Modelling Scheme Philosophy

Motivated by foundational studies concerning the modelling and analysis of complex systems we propose a scheme based on category theoretical methods and concepts [1-7]. The essence of the scheme is the development of a coherent relativistic perspective in the analysis of information structures associated with the behavior of complex systems, effected by families of partial or local information carriers. It is claimed that the appropriate specification of these families, as being capable of encoding the totality of the content, enfolded in an information structure, in a preserving fashion, necessitates the introduction of compatible families, constituting proper covering systems of information structures. In this case the partial or local coefficients instantiated by contextual information carriers may be glued together forming a coherent sheaf theoretical structure [8-10], that can be made isomorphic with the original operationally or theoretically introduced information structure. Most importantly, this philosophical stance is formalized categorically, as an instance of the adjunction concept. In the same mode of thinking, the latter may be used as a formal tool for the expression of an invariant property, underlying the noetic picturing of an information structure attached formally with a complex system as a manifold. More concretely, according

to this geometrical analogy if a complex information structure was pictured as a manifold then the information carriers would correspond to the euclidean subsets of R^n . The conceptual grounding of the scheme is interwoven with the interpretation of the adjunction between variable sets of information carriers and complex information structures, in terms of a communicative process of encoding and decoding.

Different modelling schemes for the study of systems, based on category theoretical methods, have been first developed and analyzed extensively in [11-15]. According to the author's knowledge, philosophical studies concerning the significance of category theoretical frameworks of reasoning from a variety of perspectives, appear in references [1,5,6,16-20].

2 Unit and Coint of the Adjunction

The adjunction, interpreted as a concept of parametric covariation, in Part I, is technically characterized by the notions of unit and coint [2-4,6-8,21].

For any presheaf $\mathbf{P} \in \mathbf{Sets}^{Y^{op}}$, the unit, defined as

$$\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \text{Hom}_{\mathcal{Z}}(\mathbf{A}(-), \mathbf{P} \otimes_{\mathcal{Y}} \mathbf{A})$$

has components:

$$\delta_{\mathbf{P}}(Y) : \mathbf{P}(Y) \longrightarrow \text{Hom}_{\mathcal{Z}}(\mathbf{A}(Y), \mathbf{P} \otimes_{\mathcal{Y}} \mathbf{A})$$

for each partial or local information carrier Y of \mathcal{Y} .

If we make use of the representable presheaf $y[Y]$ we obtain

$$\delta_{y[Y]} : y[Y] \rightarrow \text{Hom}_{\mathcal{Z}}(\mathbf{A}(-), y[Y] \otimes_{\mathcal{Y}} \mathbf{A})$$

Hence for each object Y of \mathcal{Y} the unit, in the case considered, corresponds to a map $\mathbf{A}(Y) \rightarrow y[Y] \otimes_{\mathcal{Y}} \mathbf{A}$. But, since,

$$y[Y] \otimes_{\mathcal{Y}} \mathbf{A} = \mathbf{L}_{\mathcal{A}\mathcal{Y}} yY \cong \mathbf{A} \circ \mathbf{G}_{y[Y]}(Y, 1_Y) = \mathbf{A}(Y)$$

the unit for the representable presheaf of information carriers is clearly an isomorphism. From the above, we conclude that the diagram below commutes:

$$\begin{array}{ccc}
 \mathcal{Y} & & \\
 \downarrow y & \searrow \mathbf{A} & \\
 \mathbf{Sets}^{\mathcal{Y}^{op}} & \xrightarrow{[-] \otimes_{\mathcal{Z}} \mathbf{A}} & \mathcal{Z}
 \end{array}$$

Thus the unit of the fundamental adjunction referring to the representable presheaf of the category of partial or local information filters, provides a structure preserving morphism $\mathbf{A}(Y) \rightarrow y[Y] \otimes_{\mathcal{Y}} \mathbf{A}$ which is an isomorphism.

On the other side, for each complex information structure Z of \mathcal{Z} the counit is

$$\epsilon_Z : \text{Hom}_{\mathcal{Z}}(\mathbf{A}(-), Z) \otimes_{\mathcal{Y}} \mathbf{A} \longrightarrow Z$$

The counit corresponds to the vertical map in the diagram below:

$$\begin{array}{ccc}
 \coprod_{v:Y' \rightarrow Y} \mathbf{A}(Y') & \begin{array}{c} \xrightarrow{\zeta} \\ \xrightarrow{\eta} \end{array} & \coprod_{(Y,p)} \mathbf{A}(Y) \rightarrow [\mathbf{R}(Z)](-) \otimes_Y \mathbf{A} \\
 & & \searrow \quad \downarrow \epsilon_Z \\
 & & Z
 \end{array}$$

The above diagram guides us to conjecture that the whole content of a complex information structure, Z in \mathcal{Z} , may be completely described in terms of the tensor product $[\mathbf{R}(Z)](-) \otimes_Y \mathbf{A}$, being the colimit in the category of elements of the [information-carriers]-variable set $\mathbf{R}(Z)$, if and only if, the counit of the established adjunction, is an isomorphism, that is, structure-preserving, 1-1 and onto. Of course, in order to substantiate the conjecture, we have to be careful to specify appropriate compatibility conditions on the overlap of the information content provided by distinct partial or local information carriers, so that, information is collated in a proper way along these filtering operational or conceptual devices, preserving simultaneously the totality of the content of the information structure they analyze. In what follows, we will realize that these specifications lead naturally to the notion of covering systems of complex information structures and effect a sheaf theoretical representation of their content.

Finally, a technically important observation, concerning the particular specification of the notion of covering systems together with the requirement

of the isomorphic nature of the counit of the adjunction, has to do with the qualification of the shaping functor introduced in Part I, or equivalently, functor of local or partial coefficients for a complex information structure, $\mathbf{A} : \mathcal{Y} \rightarrow \mathcal{Z}$, as being dense [2,8]. This qualification means that the the cocone in the category of elements of the [information-carriers]-variable set $\mathbf{R}(Z)$, that represents a complex information structure Z , in the variable environment of the category of presheaves, is universal for each Z . Intuitively, the denseness property of the shaping functor, permits an understanding of the category of complex information structures as a reflection of the category of [information carriers]-variable sets. Furthermore, it can be easily proved that, the isomorphic nature of the counit is guaranteed by the requirement of denseness of the shaping functor, since in that case, the right adjoint functor of the adjunction is characterized as full and faithful functor [2,8]. It is finally important to state that the dense characterization of the functor of local coefficients, has the desirable consequence of qualifying the category of complex information structures as complete [2,3], besides being cocomplete as part of its initial specification. This qualification, further secures that the category of complex information structures has a terminal object for the insertion of information, as well as, pullbacks responsible for the compatibility of the behavior of its structured decomposition in terms of families of local information carriers, and consequently, their integration in a coherent whole.

3 Covering Systems of a Complex Information Structure

It is instructive to start with a concise prologue, in order to emphasize the clear intuitive basis underlying the notion of covering systems. According to the proposed categorical modelling scheme, an information structure Z in \mathcal{Z} , representing the behavior of a complex system, can be understood by means of appropriate structure preserving morphisms $Y \longrightarrow Z$, interpreted as generalized coordinatizing maps. The domains of these maps are the partial or local information carriers Y in \mathcal{Y} , interpreted correspondingly, as the generators of information filtering coefficients, associated with a localizing categorical environment, or, with a structured multiple levels system of perceptual viewpoints of the content enfolded in the information structure of a complex system. More concretely, each coordinatizing map, contains the amount of information related to a specified localization context, or partial viewpoint, and thus, it represents the abstraction mechanism attached operationally or conceptually with such an information carrier. Of course, the simultaneous application of many coordinatizing maps have the potential of covering an information structure Z completely. In this case, it is legitimate to consider a suitable family of intentionally employed coordinatizing maps, as a covering system of the information structure of a complex system. Of

course, the qualification of such a structured family, as a covering system, is required to meet certain requirements, that guarantee the coherence of the categorical scheme of interpretation.

3.1 System of Prelocalizations

We start by formalizing the intuitive discussion, presented above, as follows:

A **system of prelocalizations** for a complex information structure Z in \mathcal{Z} is a subfunctor of the Hom-functor $\mathbf{R}(Z)$ of the form $\mathbf{S} : \mathcal{Y}^{op} \rightarrow \mathbf{Sets}$, namely for all Y in \mathcal{Y} it satisfies $\mathbf{S}(Y) \subseteq [\mathbf{R}(Z)](Y)$. According to this definition a system of prelocalizations for a complex information structure Z in \mathcal{Z} can be understood as a right ideal $\mathbf{S}(Y)$ of structure preserving morphisms of the form

$$\psi_Y : \mathbf{A}(Y) \longrightarrow Z, \quad Y \in \mathcal{Y}$$

such that $\langle \psi_Y : \mathbf{A}(Y) \longrightarrow Z \text{ in } \mathbf{S}(Y), \text{ and } \mathbf{A}(v) : \mathbf{A}(\acute{Y}) \rightarrow \mathbf{A}(Y) \text{ in } \mathcal{Z} \text{ for } v : \acute{Y} \rightarrow Y \text{ in } \mathcal{Y}, \text{ implies } \psi_Y \circ \mathbf{A}(v) : \mathbf{A}(\acute{Y}) \longrightarrow Z \text{ in } \mathbf{S}(Y) \rangle$.

We observe that the operational role of the Hom-functor $\mathbf{R}(Z)$ amounts to the depiction of an ideal of structure preserving morphisms, accomplishing the task of providing covers of a complex information structure by coordinatizing partial or local information carriers. In this perspective, we may characterize the coordinatizing maps $\psi_Y : \mathbf{A}(Y) \longrightarrow Z, \quad Y \in \mathcal{Y}$, in a system

of prelocalizations, as covers for the filtration of the information structure of a complex system, whereas their domains Y are the carriers of local or partial information coefficients. The above observation is equivalent to the statement that an information carrier serves as a conceptual reference frame, relative to which the information structure of a complex system is being coordinatized, in accordance to the informational specification of the corresponding localization context.

It is evident that each complex information structure can have many systems of prelocalizations, which form a partially ordered set under inclusion. We note that the minimal system is the empty one, namely $\mathbf{S}(Y) = \emptyset$ for all $Y \in \mathcal{Y}$, whereas the maximal system is the Hom-functor $\mathbf{R}(Z)$ itself. Moreover intersection of any number of systems of prelocalization is again a system of prelocalization.

Finally, it is significant to formulate the notion of generating family of a system of prelocalizations \mathbf{S} as follows: it is the set of all covers $\psi_Y : \mathbf{A}(Y) \longrightarrow Z$, $Y \in \mathcal{Y}$, such that $\psi_{Y_j} \circ \mathbf{A}(v_j) = \psi_Y$ for some v_j in \mathcal{Y} .

$$\begin{array}{ccc}
 & Z & \\
 \psi_Y \nearrow & & \nwarrow \psi_{Y_j} \\
 \mathbf{A}(Y) & \xrightarrow{\mathbf{A}(v_j)} & \mathbf{A}(Y_j)
 \end{array}$$

Equivalently we assert, that a family of covers, $[\psi_{Y_j} : \mathbf{A}(Y_j) \longrightarrow Z]$, $Y_j \in \mathcal{Y}$, generates the system of prelocalizations \mathbf{S} , if and only if, this system is the smallest among all that contain this family.

3.2 System of Localizations

The transition from a system of prelocalizations to a system of localizations, or proper covering system for a complex information structure, is the key step that guarantees the compatibility of the information content gathered in different filtering mechanisms associated with partial or local carriers of information. A proper covering system contains all the necessary and sufficient conditions for the comprehension of the content of a complex information structure, as a sheaf of partial or local coefficients associated with information carriers. The concept of sheaf expresses exactly the pasting conditions that the filtering conceptual devices have to satisfy, or else, the specification by which partial or local information concerning the structure of a complex system, can be collated together.

In order to define an informational system of localizations, it is necessary to introduce the categorical concept of pullback [1-4,6-9] in \mathcal{Z} as in the diagram below:

$$\begin{array}{ccc}
\mathbf{T} & & \\
\downarrow u & \searrow h & \\
\mathbf{A}(Y) \times_Z \mathbf{A}(Y') & \xrightarrow{\psi_{Y,Y'}} & \mathbf{A}(Y) \\
\downarrow \psi_{Y',Y} & & \downarrow \psi_Y \\
\mathbf{A}(Y') & \xrightarrow{\psi_{Y'}} & Z
\end{array}$$

The pullback of the information filtering covers $\psi_Y : \mathbf{A}(Y) \longrightarrow Z, Y \in \mathcal{Y}$ and $\psi_{Y'} : \mathbf{A}(Y') \longrightarrow Z, Y' \in \mathcal{Y}$ with common range the complex information structure Z , consists of the cover $\mathbf{A}(Y) \times_Z \mathbf{A}(Y')$ and two arrows $\psi_{Y'}$ and $\psi_{Y,Y'}$, called projections, as shown in the above diagram. The square commutes and for any object T and arrows h and g that make the outer square commute, there is a unique $u : T \longrightarrow \mathbf{A}(Y) \times_Z \mathbf{A}(Y')$ that makes the whole diagram commute. Hence we obtain the condition:

$$\psi_{Y'} \circ g = \psi_Y \circ h$$

The pullback of the information covers $\psi_Y : \mathbf{A}(Y) \longrightarrow Z, Y \in \mathcal{Y}$ and $\psi_{Y'} : \mathbf{A}(Y') \longrightarrow Z, Y' \in \mathcal{Y}$ is equivalently characterized as their fibre product, because $\mathbf{A}(Y) \times_Z \mathbf{A}(Y')$ is not the whole product $\mathbf{A}(Y) \times \mathbf{A}(Y')$ but the product taken fibre by fibre.

We notice that if ψ_Y and $\psi_{Y'}$ are 1-1, then their pullback is isomorphic with the intersection $\mathbf{A}(Y) \cap \mathbf{A}(Y')$. Then we can define the pasting map,

which is an isomorphism, as follows:

$$\Omega_{Y,\acute{Y}} : \psi_{\acute{Y}Y}(\mathbf{A}(Y) \times_Z \mathbf{A}(\acute{Y})) \longrightarrow \psi_{Y\acute{Y}}(\mathbf{A}(Y) \times_Z \mathbf{A}(\acute{Y}))$$

by putting

$$\Omega_{Y,\acute{Y}} = \psi_{Y\acute{Y}} \circ \psi_{\acute{Y}Y}^{-1}$$

Consequently we obtain the following conditions:

$$\Omega_{Y,Y} = 1_Y \quad 1_Y : \textit{identity of } Y$$

$$\Omega_{Y,\acute{Y}} \circ \Omega_{\acute{Y},\acute{Y}} = \Omega_{Y,\acute{Y}} \quad \textit{if } \mathbf{A}(Y) \cap \mathbf{A}(\acute{Y}) \cap \mathbf{A}(\acute{Y}) \neq 0$$

$$\Omega_{Y,\acute{Y}} = \Omega_{\acute{Y},Y} \quad \textit{if } \mathbf{A}(Y) \cap \mathbf{A}(\acute{Y}) \neq 0$$

The pasting map assures that $\psi_{\acute{Y}Y}(\mathbf{A}(Y) \times_Z \mathbf{A}(\acute{Y}))$ and $\psi_{Y\acute{Y}}(\mathbf{A}(Y) \times_Z \mathbf{A}(\acute{Y}))$ are going to cover the same part of a complex information structure in a compatible way.

Given a system of prelocalizations for a complex information structure $Z \in \mathcal{Z}$, we call it a localization system, if and only if, the above compatibility conditions are satisfied, and moreover, the information structure is preserved.

It is instructive to remind that the elements in a localization system for a complex information structure Z , namely the coordinatizing maps, are objects of the category of elements $\mathbf{G}(\mathbf{R}(Z), Y)$, whereas their transition functions are the morphisms of this category. This is evident, if we recall that the specification of the category of elements of $\mathbf{G}(\mathbf{R}(Z), Y)$ requires: on

the one hand, that, an object is a pair $(Y, \psi_Y : \mathbf{A}(Y) \longrightarrow Z)$, with Y in \mathcal{Y} and ψ_Y an arrow in \mathcal{Z} , that is a complex information structure preserving morphism; and on the other, that, a morphism $(\acute{Y}, \psi_{\acute{Y}}) \longrightarrow (Y, \psi_Y)$ in the category of elements is an arrow $v : \acute{Y} \longrightarrow Y$ in \mathcal{Y} , that is an information carriers structure preserving morphism, with the property that $\psi_{\acute{Y}} = \psi_Y \circ \mathbf{A}(v) : \mathbf{A}(\acute{Y}) \longrightarrow Z$; in other words, v must take the chosen cover ψ_Y in $\mathbf{G}(\mathbf{R}(Z), Y)$ back into $\psi_{\acute{Y}}$ in $\mathbf{G}(\mathbf{R}(\acute{Z}), \acute{Y})$.

The exact specification of a localization system for a complex information structure, as above, permits the comprehension of the latter as a sheaf of partial or local coefficients, associated with the variation of the information obtained in multiple localization contexts of information carriers. This follows from the fact that, the counit of the adjunction established in Part I, is an isomorphism, restricted to such an informational proper covering system, together with the property of denseness of the shaping functor, securing the existence of compatible pullbacks. In this perspective a complex information structure, may be pictured as an information manifold, obtained by pasting the $\psi_{\acute{Y}Y}(\mathbf{A}(Y) \times_Y \mathbf{A}(\acute{Y}))$ and $\psi_{Y\acute{Y}}(\mathbf{A}(Y) \times_Z \mathbf{A}(\acute{Y}))$ information covers together by the transition functions $\Omega_{Y,\acute{Y}}$.

4 Invariance in Communication of Information

The notion of functorial information communication, as established by the adjunction between preheaves of localization coefficients, associated with information filtering contexts, and, complex information structures, can be further enriched, by the formulation of a property characterizing the conditions for invariance of the information communicated via the covering systems of local or partial information carriers.

The existence of this invariance property is equivalent to a full and faithful representation of complex information structures in terms of proper covering systems, capable of encoding the whole informational content enfolded in an information structure of a complex system. We have already, specified that the intended representation is full and faithful, if and only if, the counit of the established adjunction, restricted to a proper covering system, is an isomorphism, that is, structure-preserving, 1-1 and onto.

The meaning of this representation, expresses precisely the fact that the whole information content contained in an information structure, is preserved by every family of coordinatizing maps, qualified as an informational system of localizations. The preservation property is exactly established by the counit isomorphism. Concerning the representation above, we realize that

the surjective property of the counit guarantees that the filtering mechanisms of the information carriers, being themselves objects in the category of elements, $\mathbf{G}(\mathbf{R}(Z), Y)$, cover entirely an information structure Z , whereas its injective property, guarantees that any two covers are compatible in a system of localizations. Moreover, since the counit is also a structure preserving morphism, the information structure is preserved.

We may clarify that the underlying invariance property, rooted primarily in the adjunction concept, is associated with the informational content of all different or overlapping information filtering mechanisms of the carriers, in various intentionally adopted localization contexts, and can be explicitly formulated as follows: the informational content of a structure related with the behavior of a complex system remains invariant, with respect to families of coordinatizations objectified by partial or local information carriers, if and only if, the counit of the adjunction, restricted to those families, qualified as informational localization systems, is an isomorphism.

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