PEIRCE'S CONTINUUM

A METHODOLOGICAL AND MATHEMATICAL APPROACH

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Chapter I

Genericity, Reflexivity, Modality

In this introductory monograph, we investigate Peirce's *continuum* concept from several perspectives. We stress what can be considered the main strength of Peirce's original approach to the "labyrinth of the *continuum*": (I) its central interconnection of genericity, reflexivity and modality –and subsidiary supermultitudeness, inextensibility and plasticity–, an approach which requires a careful logical treatment and which has seldom been well understood. We then show that (II) Peirce's pioneering ideas about a non-cantorian *continuum* can receive adequate partial modelling from further independent developments in XXth century mathematics. With Peirce's *continuum* concept in hand, we insist in the well-known interpretation which locates the *continuum* at the core of Peirce's system, and we proceed to show (III) some explicit uses of continuity hypotheses which pervade the skeleton of the architectonics of pragmaticism. We finally approach the elusive "proof of pragmaticism" from new perspectives, and we show that (IV) a web of crossing threads between Peirce's *continuum*, his existential graphs and his classification of the sciences becomes fundamental, helping us to understand better an evolving array of marks which can endorse the validity of Peirce's system. Following Peirce's indications on the usefulness of diagrammatic thought, we introduce an important number of figures to resume iconically some of its main trends. Secondary literature references are consistently made at the endnotes of each chapter.

I.1. Cantor's analytical object

Modern mathematics, overwhelmingly immersed into classical set theory, works with set theoretic *objects* which have only modelled part of the underlying mathematical *concepts*. A systematic identification between concept and object –coming, in part, from biased uses of Frege's abstraction principle– has limited the way to handle many mathematical concepts. In particular, the general concept of the *continuum*, when objectually transformed into Cantor's real line in modern mathematical set theory, has lost many sides of its extraordinary richness.

The cantorian real line (**R**) was constructed to solve precise and technical mathematical problems: convergence questions (Fourier series representations) in the theory of functions of real variable, and questions of local hierarchization (ordinal measure of fragments of the line) in the emerging set-theoretic topology. **R** serves to model one of the fundamental aspects of a generic *continuum*: its completeness, or "analytical saturation":

N naturals		**Z** integers		**Q** rationals		**R** reals
→ saturation processes	DIFFERENCE		PROPORTION		CONVERGENCE	
equivalence classes:		pairs of naturals		pairs of integers		Cauchy rational sequences

Figure 1.
Analytical accumulation of equivalence classes
to "saturate" the continuum in Cantor's approach

Formally reconstructed inside Zermelo-Fraenkel set theory (ZF), the number sets arise in a process of *accumulating points* inside an *actual* infinite, which begins with the naturals. The integers –as equivalence classes of the difference relation between pairs of natural numbers–, the rationals –as equivalence classes of the divisibility relation between pairs of integers–, and the reals –as equivalence classes of the convergence relation between Cauchy rational sequences– form sets of points, in which elements are added. The *summa* of those elements represents a model of the *continuum*.

Nevertheless, from a more elementary common sense perspective, without even entering yet into the composition or intrinsic properties of the *continuum*, it should be obvious that a *given* model alone (actual, determinate) cannot, in principle, capture all the richness of a *general* concept (possible, indeterminate). In fact, the pragmatic maxim hinders immediately such a pretense:

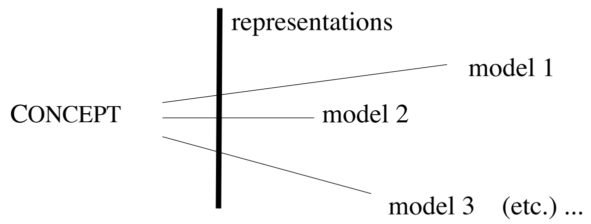


Figure 2.
Elementary use of the pragmatic maxim:
no general concept can be modelled by an object alone

The existence of multiple ways of representing and modelling should avoid any identifications of *a* mathematical concept with *a* mathematical object (something which, however, is normally and even normatively done). One of those doubtful identifications consists in the classical set theoretic formulation: *continuum* ≡ ***R***, where the idea of continuity (a general concept) is identified with the cantorian real line (a given model). Even the existence of "monstrous" models in contemporary model theory (homogeneous, saturated and universal, at will) arises with respect to *given* collections of axioms, that can only capture partially the concepts behind the axioms. It becomes fundamental, then, to distinguish the *continuum* from **R**. Another thing is that the reals help to represent –as they have effectively done so– a fundamental part of the concept of continuity.

It turns out that continuity is a *protean* concept, which –like Proteus, the mythical sea-god fabled to assume various shapes– can be modelled in several diverse ways, witnessing its extraordinary richness. More generally, as points out Saunders MacLane, one of the founders of the mathematical theory of categories,

> Mathematics is that branch of science in which the concepts are *protean*: each concept applies not to one aspect of reality, but to many[1].

MacLane's conception coincides fully with Peirce's view: mathematics moves in the unbounded realm of pure possibilities, constantly transposed into reality. Following the pragmatic maxim, the continuum (general) can only be *approached* by its different *signs* (particular models) in representational contexts. A *map* of many disguises of the *continuum* is shown in *figure 3*.

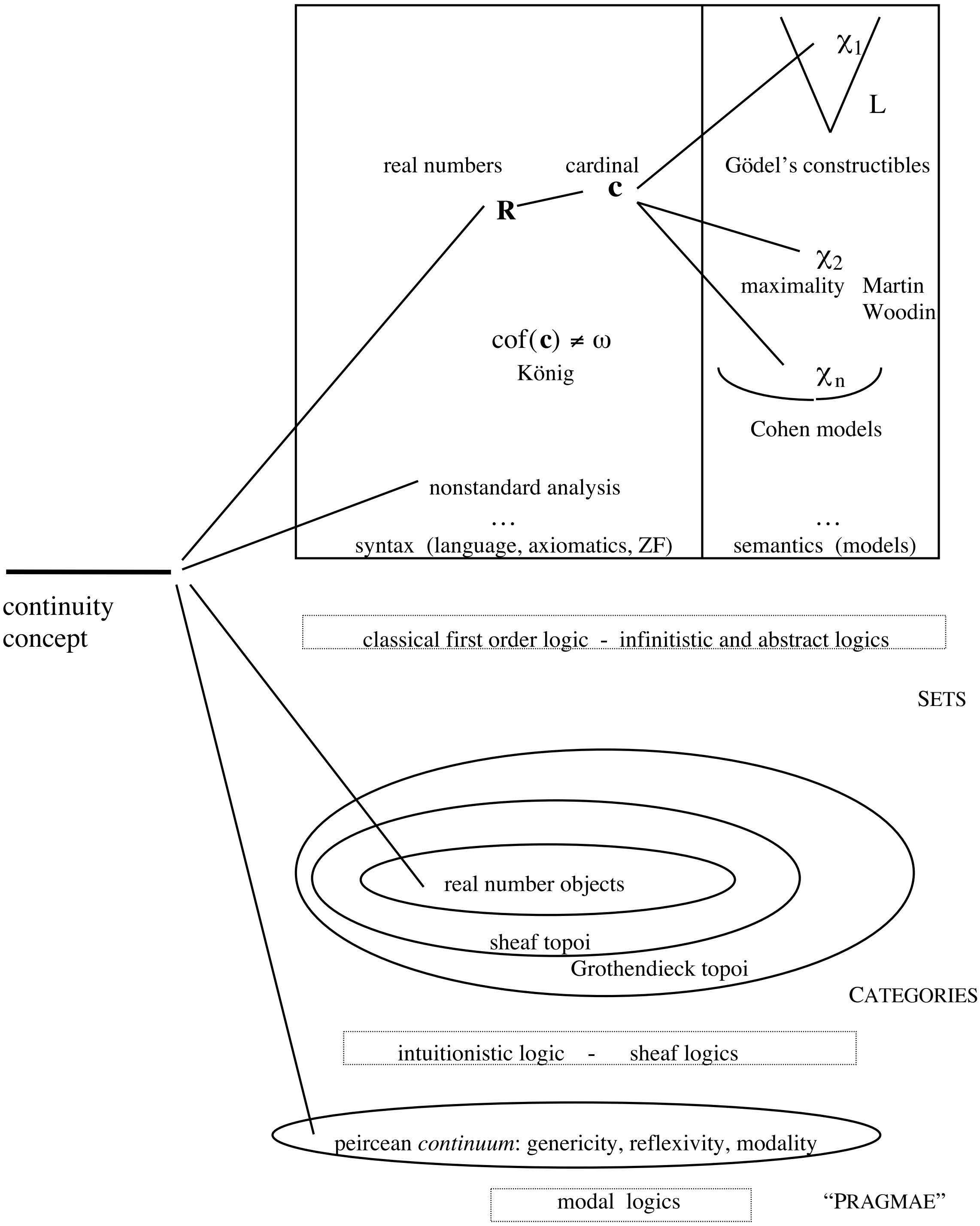


Figure 3.
Proteus: the continuum "along" the pragmatic maxim

The many shapes of the *continuum* shown in *figure 3* will be studied carefully in the second chapter. For the moment, the diagram helps to show the very particular place of Cantor's analytical object in a general outlook, and points to the situation of Peirce's *continuum*, which will be studied in what follows. It is clear that Cantor's real line **R**, which *inside* ZF plays a fundamental protean role (since its cardinal can take, there, *all* the possible forms not in contradiction with König's cofinality restriction), *outside* ZF it falls short with respect to the generic and modal richness lying in a general concept of continuity. In this sense, the cantorian real line is but a "first embryo" of continuity, as Peirce claimed, alone, in the desert.

I.2. Peirce's synthetical concept

Peirce's *continuum* is an "absolutely general" concept which, in principle, does not have to be completely objectified in just a formal context (for example, Peirce's *continuum* seems to transcend, as many great cardinal hypotheses do, the power of representation of ZF). It is a really generic concept, which intrinsically lies, in Peirce's view, inside *any* other general concept: "every general concept is, in reference to its individuals, strictly a *continuum*"[2]. Thus, Peirce's *continuum*, as a lean, "free" concept in the realm of the general and the possible, cannot be bounded by a determinated collection: "no collection of individuals could ever be adequate to the extension of a concept in general"[3]. Leaving free the determination contexts of the *continuum* –his partial "extensions"– and insisting in the intensionality of the *continuum* as a general, Peirce obtains immediately one of the profound peculiarities in his vision of the *continuum*. An original and extremely important *asymmetrization* of Frege's abstraction principle occurs: as we will further study in our second chapter, intension and extension, in multiple cases, as the one in hand, do not have to be logically equivalent.

Besides recovering the primacy of concepts over objects, Peirce insists in understanding synthetically the *continuum*, as a general whole which cannot be analytically reconstructed by an internal sum of points[4]:

> Across a line a collection of blades may come down simultaneously, and so long as the collection of blades is not so great that they merge into one another, owing to their supermultitude, they will cut the line up into as great a collection of pieces each of which will be a line, –just as completely a line as was the whole. This I say is the intuitional idea of a line with which the synthetic geometer really works, –his virtual hypothesis, whether he recognizes it or not; and I appeal to the scholars of this institution where geometry flourishes as all the world knows, to cast aside all

> analytical theories about lines, and looking at the matter from a synthetical point of view to make the mental experiment and say whether it is not true that the line refuses to be cut up into points by any discrete multitude of knives, however great.[5]

As we shall later see, this synthetical view of the *continuum* will be fully recovered by the mathematical theory of categories, in the last decades of the XX[th] century. For now, we can already record that Peirce's *continuum*, as a synthetical concept opposed to Cantor's analytical object[6], *necessarily* possesses a greater richness (indeterminate, general, vague) than the real numbers object, since –simultaneously– the conceptual reaches an ampler plurality than the objectual and the synthetical involves a wider distributed universality than the analytical.

Next diagram encompasses, in our reading, the most salient traits of Peirce's *continuum*, understood unitarily as a synthetical concept where are entangled three crucial *global* properties (genericity, reflexivity, modality), three sub-determinations of those properties (supermultitudeness, inextensibility, plasticity) and four *local* methodologies (generic relationality, vagueness logic, neighbourhood logic, *possibilia* surgery), which can weave, in local contexts, the global architecture:

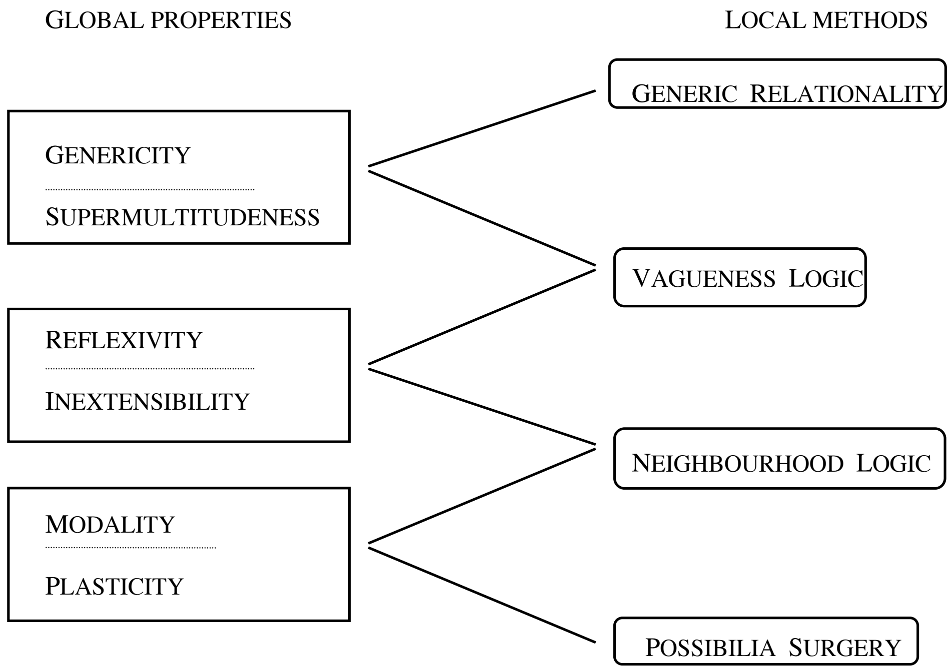


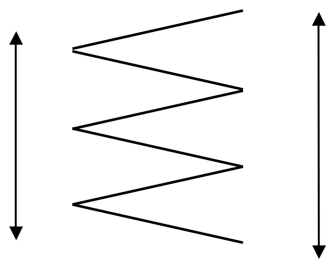
Figure 4.
The "double sigma": global and local concepts which articulate Peirce's continuum

The *double sigma* underlines some fundamental threads between global and local aspects of Peirce's *continuum* to which we will devote the rest of this chapter. The terminology tries to evoke Watson and Crick's "double helix", a double staircase of interlaced spirals where genetic information sums up. As the double helix codifies a fundamental part of the secrets of the living, the *double sigma* wishes to synthetize part of the fundamental secrets of the *continuum*.[7] A vertical reading –a pragmatic reading– of the *double sigma*, gives rise to two important programs of research, that we will call *pragmae* of the *continuum*, and whose full elucidation would need "long duration" inquiries inside our "community of researchers": the construction of a *categorical topics*, which would systematically study the global synthetic correlations between "sites" of knowledge, and the construction of a *modal geometry*, which would study the local connection methods between those sites and detect its modal "invariants".

Global Pragma

CATEGORICAL TOPICS

(versus
set-theoretic
foundations)



Local Pragma

MODAL GEOMETRY

(versus
classical
arithmetization)

Figure 5.
"Pragmae" of the continuum

As we shall see in our second chapter, XXth century mathematics, independently of Peirce, will advance in the construction of a far-reaching "categorical topics", obtaining many outstanding but somewhat isolated technical results. On the other hand, the construction of a "modal geometry" is just beginning in the last decade. One of the many legacies of Peirce's *continuum* consists in interweaving coherently the two preceding *pragmae*, finding systematically reflections of the global into the local, and vice versa. We proceed to show how Peirce's writings support the *double sigma* interpretation.

I.3. Genericity and supermultitudeness

Perhaps the most salient trait of Peirce's *continuum* is his *general* character, with all the connotations and derivations that the term includes. To adapt us a little to the more precise language of modern mathematics, we will also use the term "generic" as a substitute equivalent of "general". In Peirce, the general includes very diverse nuances, but all united under an idea of "freeness" –whatever is free of particularizing attachments, determinative, existential or actual. The general is what can live in the realm of *possibilia*, not determinate nor actual, and which opposes the particular mode of the existential. In Peirce's words,

> The idea of a general involves the idea of possible variations which no multitude of existent things could exhaust but would leave between any two not merely *many* possibilities, but possibilities absolutely beyond all multitude.[8]
>
> Generality is, indeed, an indispensable ingredient of reality; for mere individual existence or actuality without any regularity whatever is a nullity. Chaos is pure nothing.[9]

Generality –as a law or regularity beyond the merely individual, as a deep layer of reality beyond the merely named, as a basic weapon in the dispute between realism and nominalism– falls into peircean thirdness and glues naturally together with the *continuum.* Peirce recalls several times that the *continuum* can be seen as a certain form of generality:

> The continuum is a General. It is a General of a relation. Every General is a continuum vaguely defined.[10]
>
> Continuity, as generality, is inherent in potentiality, which is essentially general. (...) The original potentiality is essentially continuous, or general.[11]
>
> The possible is general, and continuity and generality are two names for the same absence of distinction of individuals.[12]
>
> A perfect continuum belongs to the genus, of a whole all whose parts without any exception whatsoever conform to one general law to which same law conform likewise all the parts of each single part. *Continuity* is thus a special kind of *generality*, or conformity to one Idea. More specifically, it is a *homogeneity*, or generality among all of a certain kind of parts of one whole. Still more specifically, the characters which are

> the same in all the parts are a certain kind of relationship of each part to all the coordinate parts; that is, it is a *regularity*.[13]

The *continuum* is thus a *general*, where all the potentialities can fall – overcoming all determinations– and where certain modes of connection between the parts and the whole (local and global) become homogenized and regularized – overcoming and melting together all individual distinctions. The generic character of Peirce's *continuum* (thirdness) is thus closely weaved with the overcoming of determinacy and actuality (secondness). In this process the threads of indetermination and chance (firstness) become essential, freeing the existent from its particular qualities in order to reach the generality of *possibilia*. For Peirce, the logic of relatives is the *natural filter* which allows to free and lean out action-reaction agents, in order to melt them in a higher general continuity, because relative logic allows to observe the individual as a "degenerate" form of relationality and the given as a degenerate form of possibility:

> Continuity is simply what generality becomes in the logic of relatives.[14]
>
> True continuity is perfect generality elevated to the mode of conception of the Logic of Relations.[15]
>
> Continuity is shown by the logic of relations to be nothing but a higher type of that which we know as generality. It is relational generality.[16]
>
> The continuum is all that is possible, in whatever dimension it be continuous. But the general or universal of ordinary logic also comprises whatever of a certain description is possible. And thus the *continuum* is that which the logic of relatives shows the *true* universal to be.[17]

Peirce's *dictum* • *continuity* = *genericity* via *relative logic* • is one of his most astonishing intuitions. In a first approach, it appears as a pretty cryptic, "occult" motto, but, as we will forcefully show in the next chapter, it really can be considered as a genial abduction, underlying the introduction of topological methods in logic and summarizing the proof (obtained in the 1990's) that many of the fundamental theorems of the logic of relatives are no more than corresponding continuity theorems in the uniform topological space of first-order logic elementary classes.

We think that this outstanding peircean abduction –clearly explicited from 1898 on and perhaps one of the firmer expressions of Peirce's logical refinement– could have been based in two previous, crucial, logical "experiments": on one side, his construction of systems of *existential graphs* (from 1896 on), where the rules of logic happen to be back-and-forth processes on the continuity of the sheet of assertion (discrete back-and-forth for the propositional calculus, and *continuous* back-and-forth for the logic of relatives –see the continual elongations of the identity line); on the other side, his neglected invention of *infinitesimal relatives* (in the never dried-out memory of 1870 on the logic of relatives)[18], which Peirce uses to reveal extremely interesting structural similarities between formal processes of differentiation (over the usual mathematical *continuum*) and operational processes of relativization (over a logical *continuum* much more general).

An immediate consequence of the genericity of the *continuum* is that the *continuum* must be *supermultitudinous*, in the sense that his size must be fully generic, and cannot be bounded by any other size actually determined[19]:

> A *supermultitudinous* collection (...) is greater than any of the single collections. (...) A supermultitudinous collection is so great that its individuals are no longer distinct from one another. (...) A supermultitudinous collection, then, is no longer *discrete*; but it is *continuous*.[20]
>
> A supermultitudinous collection sticks together by logical necessity. Its constituent individuals are no longer distinct and independent subjects. They have no existence, –no hypothetical existence–, except in their relations to one another. They are no subjects, but phrases expressive of the properties of the continuum.[21]

The supermultitudinous character of Peirce's *continuum* shows, according to Peirce, that the cantorian real line is just "the first embryon of continuity", "an incipient cohesiveness, a germinal of continuity"[22]. Nevertheless, the cardinal indetermination ($2^{\aleph_0}$) of Cantor's *continuum* inside ZF –a profound discovery of XX[th] century mathematical logic that Peirce could not imagine– shows that the cantorian model can also be considered as a valid generic candidate to capture the supermultitudeness[23] of the *continuum* (even if other generic traits of Peirce's *continuum*, in the extensible or modal realm, as we shall soon see, do not seem capable to be modelled by the cantorian real line).

In any case, from the very beginning of their investigations, Cantor and Peirce's paths are clearly opposed: while Cantor and, systematically, most of his followers, try to *bound* the *continuum*, Peirce tries to *unbound* it: to approach a supermultitudinous *continuum*, not restricted in size, truly generic in the transfinite, never totally determined. It comes then, as a most remarkable fact, that many indications of the indeterminacy of the *continuum* found at the core of contemporary cantorian set theory (free analysis of the set theoretic universe through disparate filters, using forcing techniques, with many *phenomena* possibly coexistent) seem to assure in retrospect the correction of Peirce's vision. The generality of Peirce's *continuum* implies, as we shall now see, that it cannot be reconstructed from the "particular" or the "existent", and that it must be thought in the true general realm of *possibilia*.

I.4. Reflexivity and inextensibility

One of the fundamental properties of Peirce's *continuum* consists in its *reflexivity*, a finely grained approach to Kant's conception that the *continuum* is such that any of its parts possesses in turn another part similar to the whole:

> A continuum is defined as something any part of which however small itself has parts of the same kind.[24]

We will use the term "reflexivity" for the preceding property of the *continuum* since, following a reflection principle, the whole can be reflected in *any* of its parts:

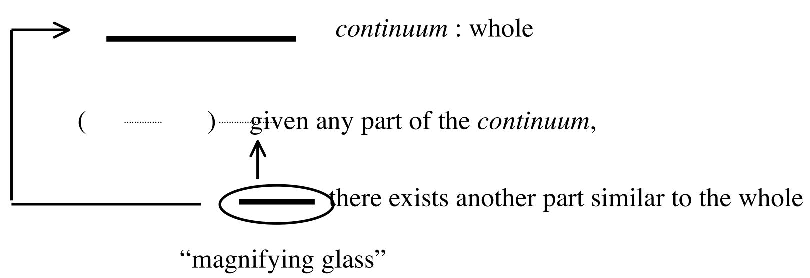


Figure 6.
The reflexivity of Peirce's continuum

As immediately infers Peirce (see next citation), reflexivity *implies* that the *continuum* cannot be composed by points, since points –not possessing other parts than themselves– cannot possess parts similar to the whole. Thus, reflexivity distinguishes at once the peircean *continuum* from the cantorian, since Cantor's real line *is* composed by points and *is not* reflexive. In Peirce's *continuum* the points disappear as actual entities (we shall see that they remain as *possibilities*) and are replaced –in actual, active-reactive secondness– by *neighbourhoods*, where the *continuum* flows:

> The result is, that we have altogether eliminated points. (...) There are no points in such a line; there is no exact boundary between any parts. (...) There is no flow in an instant. Hence, the present is not an instant. (...) When the scale of numbers, rational and irrational, is applied to a line, the numbers are insufficient for exactitude; and it is intrinsically doubtful precisely where each number is placed. But the environs of each number is called a point. Thus, a point is the hazily outlined part of the line whereon is placed a single number. When we say *is* placed, we mean *would be* placed, could the placing of the numbers be made as precise as the nature of numbers permits.[25]

We will call *inextensibility* the property which asserts that a *continuum* cannot be composed of points. As we mentioned, a *continuum*'s reflexivity implies its inextensibility (Peirce's *continuum* is reflexive, thus inextensible), or, equivalently, its extensibility implies its irreflexivity (Cantor's *continuum* is extensible, thus irreflexive). The fact that Peirce's *continuum* cannot be extensible, not being able to be captured extensionally by a sum of points, retrieves one of the basic precepts of the Parmenidean *One*, "immovable in the bonds of mighty chains", a continuous whole which cannot be broken, "nor is it divisible, since it is all alike, and there is no more of it in one place than in another, to hinder it from holding together, nor less of it, but everything is full of what is"[26].

The inextensibility of Peirce's *continuum* is closely tied to another brilliant intuition of Peirce, which states that number cannot completely codify the *continuum*:

> Number cannot possibly express continuity.[27]
>
> Lengths are not measurable by numbers, nor by limits of series of them.[28]

The impossibility to fully express the *continuum* through number grilles[29] is a *natural limitation* which shows that, in order to obtain a finer understanding of the *continuum*, the program of *classical arithmetization* of the real line (Weierstrass, Cantor) should be complemented with the new "pragma" that Peirce's writings suggest: the construction of a *modal geometrization* of the *continuum.*

I.5. Modality and plasticity

Peirce's crucial modalization of his pragmaticism can be driven, as Max Fisch has shown[30], to his late readings of the Greek Masters, at the middle of the 1880's. The Aristotelean influence –following Aristotles' use of a wide spectrum of possibilities to cover all reams of reality– weighs in Peirce's approach to the *continuum*, when he begins to present the *continuum* as a complex modal *logos*:

> A continuum is a collection of so vast a multitude that in the whole universe of possibility there is not room for them to retain their distinct identities; but they become welded into one another. Thus the continuum is all that is possible, in whatever dimension it be continuous.[31]

> You have then so crowded the field of possibility that the units of that aggregate lose their individual identity. It ceases to be a collection because it is now a continuum. (...) A truly continuous line is a line upon which there is room for any multitude of points whatsoever. Then the multitude or what corresponds to multitude of possible points, – exceeds all multitude. These points are pure possibilities. There is no such gath. On a continuous line there are not really any points at all.[32]

> It seems necessary to say that a continuum, where it *is* continuous and unbroken, contains no definite parts; that its parts are created in the act of defining them and the precise definition of them breaks the continuity. (...) Breaking grains of sand more and more will only make the sand more broken. It will not weld the grains into unbroken continuity.[33]

The great richness of real and general possibilities far exceeds the "existent" realm[34] and forms a "true" *continuum*, on which the existent must be seen as a certain type of *discontinuity.* "Existence as rupture" is another amazing peircean intuition, which anticipates by a century Weinberg's ruptures of the symmetry principle, continuity breakdowns that help to explain in contemporary physics the cosmos' evolution:

> The *zero* collection is bare, abstract, germinal possibility. The continuum is concrete, developed possibility. The whole universe of true and real possibilities forms a continuum, upon which this Universe of Actual Existence is, by virtue of the essential Secondness of Existence, a discontinuous mark.[35]

Peirce's recursive contrast between secondness and thirdness –a growing dialectics which develops its potentiality through a permanent *back-and-forth* of reflections and iterations– is also the clash between existence and being, between discontinuous mark and continuous flow, between point and neighbourhood. In Peirce's vision, while points can "exist" as discontinuous marks *defined* to anchor action-reaction number scales on the *continuum*, the "true" and steady components of the *continuum* are generic and *indefinite* neighbourhoods, interweaved in the realm of *possibilia* without actually marking its frontiers. The metaphysical process[36] which presupposes a general being prior to the emergence of existence seems to be akin to the genetic structure of Peirce's *continuum*: just like Brouwer, Peirce postulates the possibility of conceiving *previously* a global *continuum* ("higher generality"), on which marks and number systems are introduced *subsequently* to mimic locally the general *continuum* (this becomes particularly clean in Peirce's existential graphs; see our fourth chapter). As Peirce clearly suggests, the infinite breaking of grains of sand never achieves them fully merging into one another: a synthetic vision of the *continuum* (Peirce, Brouwer) has to be given previously to its analytical composition (Cantor).

Peirce's *continuum* –understood as a synthetical range where whatever is possible should be able to glue– has to be a general place (*logos*), extremely flexible, plastic, homogeneous, without irregularities:

> The perfect third is plastic, relative and continuous. Every process, and whatever is continuous, involves thirdness.[37]

> This continuum must clearly have more dimensions than a surface or even than a solid; and we will suppose it to be plastic, so that it can be deformed in all sorts of ways without the continuity and connection of parts being ever ruptured. Of this continuum the blank sheet of assertion may be imagined to be a photograph. When we find out that a proposition is true, we can place it wherever we please on the sheet, because we can imagine the original continuum, which is plastic, to be so deformed as to bring any number of propositions to any places on the sheet we may choose.[38]

The idea of continuity is the idea of a homogeneity, or sameness, which is a regularity. On the other hand, just as a continuous line is one which affords room for any multitude of points, no matter how great, so all regularity affords scope for any multitude of variant particulars; so that the idea [of] continuity is an extension of the idea of regularity. Regularity implies generality.[39]

A perfect continuum belongs to the genus, of a whole all whose parts without any exception whatsoever conform to one general law to which same law conform likewise all the parts of each single part. Continuity is thus a special kind of generality, or conformity to one Idea. More specifically, it is a homogeneity, or generality among all of a certain kind of parts of one whole. Still more specifically, the characters which are the same in all the parts are a certain kind of relationship of each part to all the coordinate parts; that is, it is a regularity. The step of specification which seems called for next, as appropriate to our purpose of defining, or logically analyzing the Idea of continuity, is that of asking ourselves what kind [of] relationship between parts it is that constitutes the regularity a continuity; and the first, and therefore doubtless the best answer for our purpose, not as the ultimate answer, but as the proximate one, is that it is the relation or relations of contiguity; for continuity is unbrokenness (whatever that may be) and this seems to imply a passage from one part to a contiguous part.[40]

Peirce's *continuum* is general, plastic, homogeneous, regular, in order to allow, in a *natural* way, the "transit" of modalities, the "fusion" of individualities, the "overlapping" of neighbourhoods. The generic idea of a *continuous flow* is present behind those transits, fusions and overlappings, ubiquitous osmotic processes that Peirce notices in the plasticity of protoplasm and human mind, and that, in a bold abduction, he lifts to a universal hypothesis:

If the laws of nature are results of evolution, this evolution must proceed according to some principle; and this principle will itself be of the nature of a law. But it must be such a law that it can evolve or develope itself. (...) Evidently it must be a tendency toward generalization, -- a generalizing tendency. But any fundamental universal tendency ought to manifest itself in nature. Where shall we look for it? We could not expect to find it in such phenomena as gravitation where the evolution has so nearly approached its ultimate limit, that nothing even simulating irregularity can be found in it. But we must search for this generalizing tendency rather in such departments of nature where we find plasticity and evolution still at work. The most plastic of all things is the human mind, and next after that comes the organic world, the world of protoplasm. Now the generalizing tendency is the great law of mind, the law of association, the law of habit taking. We also find in all active protoplasm a tendency to take habits. Hence I was led to the hypothesis that the laws of the universe have been formed under a universal tendency of all things toward generalization and habit-taking.[41]

Peirce's *continuum* –generic and supermultitudinous, reflexive and inextensible, modal and plastic– is the global conceptual *milieu* where, in a natural way, we can construct hierarchies to bound possible evolutions and local concretions of arbitrary flow notions. In the remainder of this chapter we will show how to deal with those constructions, completing thus an introductory overview to the "double sigma" interpretation of Peirce's *continuum* (*figure 4*). We will study some of the local methods that Peirce devised to begin to control the specific "passages" of genericity, reflexivity and modality.

I.6. The local methods

True discoverer of all the potentiality lying in the logic of relatives[42], Peirce applies the strength of that logical lens to the problem of approaching locally the *continuum.* Turning to the genericity of the *continuum*, Peirce notices that the "mode of connection" of the parts must be understood in full generality, involving a genuine triadic relation, and he opens thus the way to a study of *generic triadic relations*, closely tied with "general modes" of smoothness and contiguity:

> My notion of the essential character of a perfect continuum is the absolute generality with which two rules hold good, first, that every part has parts; and second, that every sufficiently small part has the same mode of immediate connection with others as every other has.[43]
>
> No perfect continuum can be defined by a dyadic relation. But if we take instead a triadic relation, and say A is r to B for C, say, to fix our ideas, that proceeding from A in a particular way, say to the right, you reach B before C, it is quite evident that a continuum will result like a self-returning line with no discontinuity whatever...[44]
>
> The attraction of one particle for another acts through continuous Time and Space, both of which are of triadic constitution. (...) The dyadic action is not the whole action; and the whole action is, in a way, triadic.[45]

These assertions show that Peirce is trying to find fitting reflections of the global into the local: the *continuum* –which in its "perfect generality" is one of the most achieved global forms of thirdness– must also embody a genuinely triadic mode[46] of connection in the constitution of its local fragments.

Peirce's *continuum*, as a general, is indeterminate. Along what we could call indetermination "fibers", the general reacts antithetically with the "vague":

> Logicians have too much neglected the study of vagueness, not suspecting the important part it plays in mathematical thought. It is the antithetical analogue of generality. A sign is objectively general, in so far as, leaving its effective interpretation indeterminate, it surrenders to the interpreter the right of completing the determination for himself. "Man is mortal." "What man?" "Any man you like." A sign is objectively vague, in so far as, leaving its interpretation more or less indeterminate, it reserves for some other possible sign or experience the function of completing the determination. "This month," says the almanac-oracle, "a great event is to happen." "What event?" "Oh, we shall see. The almanac doesn't tell that."[47]

We refer to next figure for a visual image of the situation. To an important degree, the study of generality can be seen as the study of the universal quantifier ("*any* man"), while the study of vagueness is the study of the existential quantifier ("*a* great event"). As we will see in our second chapter, an explicit adjunction, or evolving antithesis, between genericity (∀) and vagueness (∃) was to be found, and precisely studied, by another great american mathematician in the 1960's.

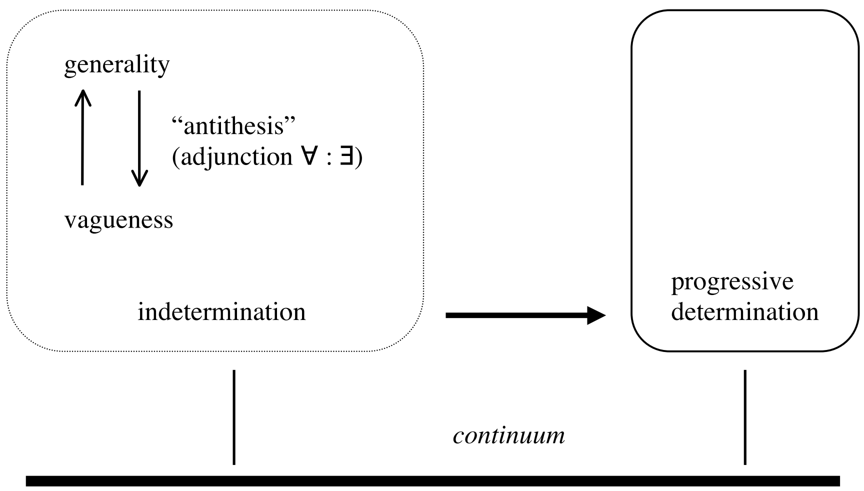


Figure 7.
Generality-vagueness "adjunction"
in the indeterminate "fibers" of the continuum

Peirce's *logic of vagueness*[48] hopes to control the transit of the indefinite to the definite, of the indeterminate to the determinate, and to study some intermediate borders[49] in processes of relative determination. Prior to this horizontal control, nevertheless, Peirce discovered the basic vertical antithesis • *genericity* vs. *vagueness* • whose partial resolutions were to pave the way to the construction of intermediate logical systems. The "antithesis", when applied locally to the *continuum*, weaves closely a scheme of general connexion modes, *naturally intermediate*:

> A point of a surface may be in a region of that surface, or out of it, or on its boundary. This gives us an indirect and vague conception of an intermediary between affirmation and denial in general, and consequently of an intermediate, or nascent state, between determination and indetermination. There must be a similar intermediacy between generality and vagueness.[50]

Mathematical logic in the XXth century would show that the *natural* logic associated to the connecting modes of the *continuum* is really an intermediate logic – the intuitionistic logic– in which the principle of excluded middle does *not* hold. It is thus amazing that Peirce –following general paths in his architectonics, very distant from the technical demands that underlie intuitionistic constructive threads– could have been able to predict that an adequate logic for the *continuum* would have to abandon, in fact, the law of excluded middle:

> If we are to accept the common sense idea of continuity (after correcting its vagueness and fixing it to mean something) we must either say that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points. The principle of excluded middle only applies to an individual (for it is not true that "Any man is wise" nor that "Any man is not wise"). But places, being mere possibles without actual existence, are not individuals. Hence a point or indivisible place really does not exist unless there actually be something there to mark it, which, if there is, interrupts the continuity.[51]
>
> I must show that the *will be's*, the actually *is's*, and the *have beens* are not the sum of the reals. They only cover actuality. There are besides *would be's* and *can be's* that are real. The distinction is that the *actual* is subject both to the principles of contradiction and of excluded middle; and in *one* way so are the *would be's* and *can be's*. In *that* way a *would be* is but the negation of a *can be* and conversely. But in another way a *would be* is not subject to the principle of excluded middle; both *would be X* and *would be not X* may be false. And in this latter way a *can be* may be defined as that which is not subject to the principle of contradiction. On the contrary, if of anything it is only true that it *can be X* it *can be not X* as well.[52]

In these two quotes, Peirce points out that the logic of actuality can be approached by usual classical logic, but that the "true" logic of continuity (to be applied to the dynamical flow of potential sites and not to the static condition of points) is a logic where the principle of excluded middle fails. In his rather difficult language of "vague" modalities ("can be": $\Diamond$; "would be": $\neg\Diamond$) , Peirce also relates generality and necessity (forms of thirdness), as well as vagueness and possibility (forms of firstness), and tries to *characterize* logically the former as *failures of distribution* of the excluded middle, as well as the latter as *failures of distribution* of the contradiction principle[53]:

> The general might be defined as that to which the principle of excluded middle does not apply. A triangle in general is not isosceles nor equilateral; nor is a triangle in general scalene. The vague might be defined as that to which the principle of contradiction does not apply. For it is false neither that an animal (in a vague sense) is male, nor that an animal is female.[54]

Failure of Excluded Middle:	Failure of Contradiction Principle:
$p \vee \neg p$ fails for the general ($\forall$) and for the necessary ($\Box$)	$\neg (p \wedge \neg p)$ fails for the vague ($\exists$) and for the possible ($\Diamond$)
$\nvDash \forall x P \vee \forall x \neg P$	$\nvDash \neg (\exists x P \wedge \exists x \neg P)$
$\nvDash \Box p \vee \Box \neg p$	$\nvDash \neg (\Diamond p \wedge \Diamond \neg p)$

Figure 8.
Generality and Vagueness do not distribute

In our next quote we will see how Peirce, after analyzing a situation in all its *possible generality*, once again extrapolates his logical acumen to a cosmological hypothesis. These risky and fascinating abductions are based, in our view, in a *double continuity hypothesis*: the hypothesis that the *logical continuum* –composed by relative logic and its intermediate layers– is a true reflection of the cosmos' *continuum*, and the hypothesis that *free*, *generic* assertions behave similarly between local and global structural forms of the *continuum*:

> The evolution of forms begins or, at any rate, has for an early stage of it, a vague potentiality; and that either is or is followed by a continuum of forms having a multitude of dimensions too great for the individual dimensions to be distinct. It must be by a contraction of the vagueness of that potentiality of everything in general, but of nothing in particular, that the world of forms comes about.[55]
>
> All that I have been saying about the beginnings of creation seems wildly confused enough. Now let me give you such slight indication, as brevity permits, of the clue to which I trust to guide us through the maze. Let the clean blackboard be a sort of diagram of the original vague potentiality, or at any rate of some early stage of its determination. This is something more than a figure of speech; for after all continuity is generality. This blackboard is a continuum of two dimensions, while that which it stands for is a continuum of some indefinite multitude of dimensions. This blackboard is a continuum of possible points; while that is a continuum of possible dimensions of quality, or is a continuum of possible dimensions of a continuum of possible dimensions of quality, or something of that sort. There are no points on this blackboard. There are no dimensions in that continuum. I draw a chalk line on the board. This discontinuity is one of those brute acts by which alone the original vagueness could have made a step towards definiteness. There is a certain element of continuity in this line. Where did this continuity come from? It is nothing but the original continuity of the blackboard which makes everything upon it continuous.[56]

A generic *continuum* is always present in the universe, reflected in multiple layers (single *continuum* of qualitative possibilities – line in the blackboard) and "meta-layers" (double *continuum* of qualitative possibilities – blackboard). Through acts of "brute force" are then produced breaks on the *continuum* which allow to "mark" differences: secondness, existence, discreteness, emerge all as *ruptures* of the real, the third, the continuous. Vagueness, indetermination, amalgamation, present in a "primitive" *continuum* (the Parmenidean "One"), evolve towards a logic of identity, more and more determined, capable of recording differences *by means of* successive breaks, ruptures, discontinuities. Since the evolution is not absolute, but contextual, nor achievable, but partial, the counterpoint between a continuous ground and discontinuity peaks becomes *saturated* only in certain given contexts. Changing flows of the living, or approaching visions of the world, in other contexts, acquire a completely new dynamics. A peircean model for the development of the universe can be seen as an evolving spiral, which extends in a three-dimensional *continuum* and which crosses diverse conceptual cylinders that symbolize *both* the natural and the cultural world: in each crossing, in each "mark", a new height and a new location allow to construct a new perspective, from which the unlimited peircean *semeiosis* can transform herself.

Peirce's *continuum* is formed by superposed "real" environments and neighbourhoods –modes of fusion and connection of the *possibilia*–. On that *continuum* "ideal" points are marked –cuts and discontinuities of the actual– only to construct contrasting scales and to facilitate the "calculus". An apparent oddity, which ties the real with the possible and the ideal with the actual, is one of the radical stakes of peircean philosophy[57]. Indeed, the actual, the given, the present, the instant, are no more than *ideal limits*: limits of possibility neighbourhoods which contain those actuality marks, those points impossible to be drawn, those fleeting presents, those impalpable instants.

Accordingly, Peirce insists that the *continuum* must be studied –in a coherent approach with its inextensibility– by means of a *neighbourhood logic*: an intermediate logic which would study the connecting modes of *environments* of the real, a non-classical logic which would go beyond *punctual* "positive assertion and negation":

> I have long felt that it is a serious defect in existing logic that it takes no heed of the *limit* between two realms. I do not say that the Principle of Excluded Middle is downright *false*; but I *do* say that in every field of thought whatsoever there is an intermediate ground between *positive assertion* and *positive negation* which is just as Real as they.[58]

> A *continuum* (such as time and space actually are) is defined as something any part of which however small itself has parts of the same kind. Every part of a surface is a surface, and every part of a line is a line. The point of time or space is nothing but the ideal limit towards which we approach indefinitely close without ever reaching it in dividing time or space. To assert that something is true of a point is only to say that it is true of times and spaces however small or else that it is more and more nearly true the smaller the time or space and as little as we please from being true of a sufficiently small interval. (...) And so nothing is true of a point which is not at least on the limit of what is true for spaces and times.[59]

> A drop of ink has fallen upon the paper and I have walled it round. Now every point of the area within the walls is either black or white; and no point is both black and white. That is plain. The black is, however, all in one spot or blot; it is within bounds. There is a line of demarcation between the black and the white. Now I ask about the points of this line, are they black or white? Why one more than the other? Are they (A) both black and white or (B) neither black nor white? Why A more than B, or B more than A? It is certainly true, First, that every point of the area is either black or white, Second, that no point is both black and white, Third, that the points of the boundary are no more white than black, and no more black than white. The logical conclusion from these three propositions is that the points of the boundary do not exist. That is, they do not exist in such a sense as to have entirely determinate characters attributed to them for such reasons as have operated to produce the above premises. This leaves us to reflect that it is only as they are connected together into a continuous surface that the points are colored; taken singly, they have no color, and are neither black nor white, none of them. Let us then try putting "neighboring part" for point. Every part of the surface is either black or white. No part is both black and white. The parts on the boundary are no more white than black, and no more black than white. The conclusion is that the parts near the boundary are half black and half white. This, however (owing to the curvature of the boundary), is not exactly true unless we mean the parts in the immediate neighborhood of the boundary. These are the parts we have described. They are the parts which must be considered if we attempt to state the properties at precise points of a surface, these points being considered, as they must be, in their connection of continuity. One begins to see that the phrase "immediate neighborhood," which at first blush strikes one as almost a contradiction in terms, is, after all, a very happy one.[60]

Peirce's arguments show that talking of "points" in the boundary of the ink drop is just an "ideal" postulate. There exist *really* only colored environments in the paper, of three specific kinds: black, white, or black-*and*-white neighbourhoods. Boundary "points" are characterized as ideal entities which can only be approached by neighbourhoods of the third kind. Thus, neighbourhood logic –or "continuous coloring" logic– embodies elementary forms of thirdness and triadicity[61], and discards immediately the law of excluded middle. It should not come as a surprise, then, that Peirce, in attempts to construct triadic connectives[62], would become the first modern logician to construct truth-tables with intermediate truth-values.

In Peirce's *continuum* the neighbourhoods are *possibilia* environments[63], where a supermultitude of potential "points" accumulate. In many approaches to Peirce's *continuum*, those *possibilia* have been described as infinitesimal monads: around an actual mark on the continuous line stands a supermultitudinous myriad of infinitesimals[64]. It will be of prime concern to construct a "local surgery" in the geometry of those possibility realms[65], which should involve similar techniques to Whitney's surgery techniques in differential topology, and with which germs of possibility could be glued[66] and deployed simultaneously. That *possibilia surgery* – still fully to be developed, but implicit in Peirce's approach (for example, pretty clear in the erasure and deiteration processes in the existential graphs)– should be able to naturally interweave with Thom's cobordism techniques (a "generic cobordism" should be part of a generic third[67]) and with Thom's call on an "archetypical" *continuum* –a "topos" qualitatively homogeneous– similar in many ways to Peirce's *continuum*.

We think that Peirce's *continuum* hooks up perfectly with Leibniz' "maximal principle", according to which the world articulates along the simplest hypothesis and the richest phenomena. Peirce's *continuum* covers, in fact, a huge phenomenical range, while it articulates only three simple concepts –genericity, reflexivity, modality– from which follows a wide spectrum of global and local characteristics.

In the next chapter, after a contrast (induction) of several alternative models for the *continuum* proposed in XX[th] century mathematics, we proceed to decant some of the "simple" mathematical hypothesis (abduction) which underlie those models, and whose eventual formal unification (deduction) could help to construct new approaches to Peirce's *continuum*.

References.

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[CP]: C.S. Peirce, *Collected Papers*, 8 vols. (eds. Hartshorne, Weiss & Burks), Bristol: Thoemmes Press, 1998 (new reprint of Harvard University Press original edition, 1931-1958); electronic edition (cd-rom): Intelex Corporation, 1992. [W]: C.S. Peirce, *Writings (A Chronological Edition)*, 6 vols. (eds. Houser et.al., in progress), Bloomington: Indiana University Press, 1982-2000. [EP]: C.S. Peirce, *The Essential Peirce*, 2 vols. (eds. Kloesel, Houser, De Tienne), Bloomington: Indiana University Press, 1992- 1998. [NEM]: C.S. Peirce, *The New Elements of Mathematics*, 4 vols. (ed. Eisele), The Hague: Mouton, 1976. [HP]: C.S. Peirce, *Historical Perspectives on Peirce's Logic of Science*, 2 vols. (ed. Eisele), The Hague: Mouton, 1985. [RLT]: C.S. Peirce, *Reasoning and the Logic of Things* (ed. Ketner), Cambridge: Harvard University Press, 1992. [PPM]: C.S. Peirce, *Pragmatism as a Principle and Method of Right Thinking* (ed. Turrisi), Albany: State University of New York Press, 1997.

In order to provide some attachments in Peirce's evolving thought, each reference to Peirce provides, between square brackets [ ], a *year* (if known) when the text was written and a *place* where the text can be found, between Peirce´s published writings. For example, [1903; EP 2.147] refers to a text from 1903, to be found in the second volume of the *Essential Peirce*, page 147, or [c.1896; CP 1.417] refers to a text close to 1896, to be found in the first volume of the *Collected Papers*, paragraph 417.

[1] Saunders MacLane, "Is Mathias an ontologist?", en: Haim Judah, Winfried Just, Hugh Woodin (comps.), *Set Theory and the Continuum*, New York: Springer, 1992, p. 120.

[2] "Some amazing mazes" [1908; CP 4.642].

[3] "Consequences of critical common-sensism" [1905; 5.526].

[4] An important distinction between "topicists" and "analysts" is drawn by Peirce in Ms. 137 (1904): while analysts build up the continuum on points, inversely, topicists drop down points from the continuum. See B. Noble, "Peirce's Definitions of Continuity and the Concept of Possibility", *Transactions of the Charles S. Peirce Society* XXV (1989), 149-174, p.151.

[5] "Multitude and continuity" [c.1895-1900; NEM 3.96].

[6] V. Potter and P. Shields, "Peirce's Definitions of Continuity", *Transactions of the Charles S. Peirce Society* XIII (1977), 20-34, set four "main periods" for the eventual development of Peirce's *continuum*: pre-Cantorian (before 1884), Cantorian (1884-1894), Kantistic (1895-1908), post-Cantorian (1908-1911). It is not our task in this chapter to redress a historical understanding of the development lines of Peirce's thought, but the triple emphasis of Potter and Shields around Cantor seems excessive. As we will show, Peirce's *continuum* is by and large a non-cantorian construction articulated around generals, reflections and *possibilia* inexistent in Cantor's real line. A more faithful historical approach seems to be J. Dauben, "Charles S. Peirce, Evolutionary Pragmatism and The History of Science", *Centaurus* 38 (1996), 22-82, where Dauben shows that Peirce's interest in infinity was prompted by *logical* questions (syllogisms and modalities, independently of Cantor) and not by *mathematical* considerations. As Dauben shows, a good divergence standpoint in Peirce's and Cantor's approaches can be seen in their disparate understanding of the theorem (found independently by both) that the power of a multitude is greater than that multitude. While Cantor constructs then the hierarchy of alephs to try to track *down* the power cardinals, Peirce unleashes *up* the supermultitudeness of the *continuum*: it cannot possess any determinate multitude since then it could be surpassed by the power of that multitude.

[7] Here we follow the suggestion of R. Dipert, "Peirce's Underestimated Place in the History of Logic: A Response to Quine", in: K. Ketner (ed.), *Peirce and Contemporary Thought: Philosophical Inquiries*, New York: Fordham University Press, 1995, 32-58, who claims that a study of Peirce's writings on sets "would return us to where earnest thought on the subject dropped off" (p. 50). In fact, we hope to prove that reading Peirce *independently* of Cantor –as we continue to do in the remainder of this chapter– will return us to a wider landscape than the one provided by Cantor's "first embryo" of continuity.

[8] "Lectures on Pragmatism" [1903; CP 5.103].

[9] "What Pragmatism Is" [1905; CP 5.431].

[10] "Letter to E.H. Moore" [1902; NEM 3.925].

[11] "Detached Ideas on Vitally Important Topics" [1908; CP 6.204-205].

[12] "Multitude and Number" [1897; CP 4.172].

[13] "Some Amazing Mazes" - "Supplement" [1908; CP 7.535 note 6].

[14] "What Pragmatism Is" [1905; CP 5.436]. Another similar expression has been noticed in G. Locke, "Peirce's Metaphysics: Evolution, Synechism, and the Mathematical Conception of the Continuum", *Transactions of the Charles S. Peirce Society* XXXVI (2000), 133-147. In Ms.4.5 Locke reckons Peirce's statement: "corresponding to generality in nonrelative logic is continuity in relative logic". The correspondence [generality : higher-order logic :: continuity : first-order logic] is yet to be fully explored, but we provide some clues in our second chapter.

[15] "Consequences of Critical Common-Sensism" [1905; CP 5.528].

[16] "Detached Ideas on Vitally Important Topics" [1908; CP 6.190].

[17] "Detached Ideas Continued and the Dispute Between Nominalists and Realists" [1898; NEM 4.343].

[18] "Description of a Notation for the Logic of Relatives" [1870; W 2.359-429]. The "infinitesimal relatives" appear at pages 395-408. Considered by D. Merrill as "elaborate and obscure mathematical analogies" [W 2.xlviii], the infinitesimal relatives were introduced by Peirce to show that certain identities in first-order logic could be seen as differential "marks" of continuous processes. Our poor understanding of Peirce's ideas should be rather called "obscure"; it is hoped that time will show that

Peirce's analogies were difficult but magnificent anticipations: his infinitesimal relatives seem just close enough to some of the "quantale" manipulations now advanced in differential geometry. T. Herron, "C.S. Peirce's Theories of Infinitesimals", *Transactions of the Charles S. Peirce Society* XXXIII (1997), 590-645, draws also attention, in his second appendix, towards what could be obtained from a good understanding of Peirce's infinitesimal relatives.

[19] H. Putnam, "Peirce's Continuum", in Ketner, op.cit., 1-22, tries to describe the supermultitudeness of Peirce's *continuum* through the "cardinal Ω of all sets" (p.11). In our view, this great cardinal hypothesis would still be short of the truly supermultitudeness of the *continuum*, which should not be reached by any given cardinal (however big). What is again in hand is that Peirce's *continuum* should not to be reachable in the cumulative cantorian set-theoretic hierarchy (restricted to V or *extended* through great cardinals). Closer to Peirce's spirit would seem König's efforts to prove that the *continuum* is *not* an aleph: in König's words (1905), "the second-number class cannot be considered to be a complete set" (cited in R. Dipert, "Peirce's Philosophical Conception of Sets", in: H. Houser, D. Roberts. J. Van Evra (eds.), *Studies in the Logic of Charles Sanders Peirce*, Bloomington: Indiana University Press, 1997, 53-76). On the other hand, the essential indeterminacy of the *continuum* (or incompleteness in König's sense) has been thoroughly emphasized by Putnam, who finds the "key" to Peirce's *continuum* in the "possibility of repeated division which can never be exhausted in any possible world in which one can complete abnumerably infinite processes" (Putnam, op.cit., p. 17).

[20] "Multitude and Continuity" [c.1897; NE 3.86-87].

[21] *Ibid*. [c.1897; NE 3.95].

[22] *Ibid*. [c.1897; NE 3.88].

[23] However, this assertion must be treated with caution, since in fact most of the efforts of set theory researchers are directed to find *additional natural axioms* to ZF, in order to *fix and determine* the size of the *continuum* in a given rank of aleph's hierarchy (natural minimality: $\aleph_1$: Gödel; natural maximality: $\aleph_2$: Martin, Woodin). The indetermination of $2^{\aleph_0}$ is not well appreciated by the specialists, who consider that the incompleteness of ZF must be *repaired*. In any of the "normative" responses given to the cardinal size of the *continuum*, it ceases immediately to be supermultitudinous since the additional axioms *force* it to adjust in a determinate level of a hierarchy.

[24] "The Conception of Time Essential in Logic" [1873; W 3.103]. Peirce, like Kant, would confuse, for some time, the reflexivity of the continuum with its infinite divisibility; see, for example: "I have termed the property of infinite intermediety, or divisibility, the Kanticity of a series", in: "Grand Logic" [1893; CP 4.121], or "The Kanticity is having a point between any two points", in: "Century Dictionary" [1898; CP 6.166]. Nevertheless, Peirce would revise since 1900 his understanding of the "kanticity", or reflexivity, of the *continuum*, understanding fully the complexity of 1873's assertion. See, for example, his *mea culpa*: "Further study of the subject has proved that this definition is wrong. It involves a misunderstanding of Kant's definition which he himself likewise fell into. Namely he defines a continuum as that all of whose parts have parts of the same kind. He himself, and I after him, understood that to mean infinite divisibility, which plainly is not what constitutes continuity since the series of rational fractional values is infinitely divisible but is not by anybody regarded as continuous. Kant's real definition implies that a continuous line contains no points", in: "Marginal Note" [1903; CP 6.168], or also: "The above conception of a line leads to a definition of continuity very similar to that of Kant. Although Kant confuses continuity with infinite divisibility, yet it is noticeable that he always defines a continuum as that of which every part (not every *echter Theil*) has itself parts. This is a very different thing from infinite divisibility, since it implies that the continuum is not composed of points", in: "Letter to the Editor of Science" [1900; CP 3.569].

[25] "On Continuous Series and the Infinitesimal" [NEM 3.126-127].

[26] "The Way of Truth", in *Parmenide's Poem* (John Burnet ed., R.P. 118).

[27] "Multitude and Continuity" [c.1897; NEM 3.93].

[28] "On Continuous Series and the Infinitesimal" [NEM 3.127].

[29] The impossibility to express the *continuum* through number grilles is also one of the basic characteristics of Brouwer's intuition of the *continuum*, as we shall see in our second chapter.

[30] M. Fisch, "Peirce's Arisbe: The Greek Influence in His Later Philosophy", in: M. Fisch, *Peirce, Semeiotic and Pragmatism* (eds. Ketner, Kloesel), Bloomington: Indiana University Press, 1986.

"From Epicurus's chance, for example, Peirce moved to the chance and spontaneity of Aristotle, and in general to Aristotle's logical and physical modalities in relation to his own categories", *ibid.*, p. 232.

[31] "Cambridge Lectures" [1898; RLT 160].

[32] "Lowell Lectures" [1903; NEM 3.388].

[33] "Marginal Note" [1903; CP 6.168]. On the very important paragraph 6.168, see Fisch (op.cit), p. 246, note 18.

[34] In Putnam's words, underlining the modal key to Peirce's *continuum*, "possibility *intrinsically* outruns actuality, not just because of the finiteness of human powers" (Putnam, op.cit., p. 19).

[35] "Detached Ideas Continued and the Dispute Between Nominalists and Realists" [1898; NE 4.345].

[36] We should not fear to speak of "metaphysics" in a mainly scientific approach! A fundamental part of Peirce's program was to construct a "mathematical metaphysics" [CP 6.213] –fulfilling thus Leibniz' dream–, where concepts could be cleaned up of the hotchpotch which obscured them (systematic uses of the pragmatic maxim), but where one could turn again, with renewed vigor and naturalness, to the great open questions of Greek and scholastic philosophy.

[37] "One, Two, Three: An Evolutionist Speculation" [1886; W 5.301].

[38] "Lowell Lectures" [1903; CP 4.512]. The "assertion sheet" refers to Peirce's existential graphs: they constitute a local and technical model, utterly precise, where many of the more daring and "speculative" Peirce assertions on the *continuum* incarnate. As some Peirce scholars have *shown*, and as we hope to *prove* in our fourth chapter, the existential graphs are the masterpiece of Peirce's logic ("My *chef d'oeuvre*", in: "Letter to Jourdain" [1908], cited in D. Roberts, *The Existential Graphs of Charles S. Peirce*, The Hague: Mouton, 1973, p. 110). To continue ignoring the existential graphs is a true contradiction with Peirce's architectonical and logical thought, thoroughly reflected in the outstanding architectonics of the graphs.

[39] "On Topical Geometry, in General" [CP 7.535].

[40] "Supplement" [1908; CP 7.535, note 6].

[41] "Cambridge Lectures" - "Habit" [1898; CP 7.515].

[42] Peirce recognized De Morgan as one of his spiritual fathers ("my master, Augustus De Morgan", in: "A Syllabus of Certain Topics of Logic" [1903; CP 3.574, note 2]), but the true vision and the development of the logic of relatives are due to the extraordinary Peirce memoirs of the years 1870-1885.

[43] "Some Amazing Mazes" - "Addition" [1908; CP 4.642].

[44] "The Logic of Events" [1898; CP 6.188].

[45] "Some Amazing Mazes" - "Fourth Curiosity" [c.1909; CP 6.330].

[46] C. Eisele, "The Problem of Mathematical Continuity" (in C. Eisele, *Studies in the Scientific and Mathematical Philosophy of C.S. Peirce*, The Hague: Mouton, 1979, 208-215) stressed the fundamental idea that a study of Peirce's logic of continuity should involve, as a first approximation, a 3-valued logic.

[47] "Consequences of Critical Common-Sensism" [c.1905; CP 5.505].

[48] For an extensive study of Peirce's vagueness, see J.E. Brock, *C.S. Peirce's Logic of Vagueness*, Ph.D. Thesis, Urbana: University of Illinois, 1969.

[49] For a nice description of Peirce's relations between continuity and vagueness, see R. Fabbrichesi Leo, *Continuità e vaghezza*, Milano: CUEM, 2001, pp.140-149.

[50] "Issues of Pragmaticism" [1905; CP 5.450].

[51] "Marginal Note" [1903; CP 6.168].

[52] "Letter to Paul Carus" [c.1910; CP 8.216].

[53] Another description of this situation can be found in B. Noble, op.cit., p.170, where possibilities, or "may-be's", fail the principle of contradiction, and continuities, or "would-be's", fail the principle of excluded middle.

[54] "Consequences of Critical Common-Sensism" [c.1905; CP 5.505].

[55] "The Logic of Events" [1898; CP 6.196].

[56] *Ibid.* [1898; CP 6.203].

[57] Peirce's weavings between possibility and realism, and actuality and idealism –*incarnated technically in the continuum*– constitute, if not a complete renewal of philosophy, at least a fresh coming back to the Greek masters and the scholastics, which revives the "transcendental" outlook of German idealism.

[58] "Letter to William James" [1909; manuscript cited in Max Fisch, op.cit., p. 180].

[59] "The Conception of Time Essential in Logic" [1873; W 3.106]. In another draft of the same text, Peirce affirms that "a point of time differs in no respect from an interval, except that it is the ideal limit. And if nothing is present for any length of time, nothing is present in an instant". [1873; W 3.103].

[60] "Grand Logic" [1893; CP 4.127].

[61] R. Lane, "Peirce's Triadic Logic Revisited", *Transactions of the Charles S. Peirce Society* XXXV (1999), 284-311, has forcefully shown that Peirce's triadic logic has to be understood as a logic concerning the "ink blot": a natural *continuity logic*, far away from arbitrary polyvalent formal manipulations.

[62] See M. Fisch, "Peirce's Triadic Logic", op.cit., pp. 171-183. Peirce's manuscripts are from1909. The manuscripts were not published before Fisch, and therefore they did not have any influence in the development of many-valued logics.

[63] Our neighbourhoods, or *possibilia* environments, correspond to Putnam's "point parts" (Putnam, op.cit. pp.7-8), and can be seen also as infinitesimal monads.

[64] For a thorough account of Peirce's infinitesimals, see T. Herron, op.cit. C. Hausman, "Infinitesimals as Origins of Evolution: Comments Prompted by Timothy Herron and Hilary Putnam on Peirce's Synechism and Infinitesimals", *Transactions of the Charles S. Peirce Society* XXXIV (1998), 627-640, forcefully shows that *possibilia* constitute privileged *loci* of branching which support Peirce's spontaneity and creativity sparks. In our last chapter we try to show that the triadic branching of the classification of sciences, understood in the *continuous environment* of gamma existential graphs, can be seen just as such a creativity spark.

[65] The eventual interest of a "geometry of *possibilia*" for the understanding of Peirce's *continuum* is also supported by the historical evidence lying behind many of the geometrical motivations interweaved in Peirce's approach to the *continuum.* M. Murphey, *The Development of Peirce's Philosophy*, Cambridge: Harvard University Press, 1961, chapters VIII-XI, and C. Eisele, "Mathematical Exactitude in C.S. Peirce's Doctrine of Exact Philosophy" (in: K. Ketner (ed.), *Proceedings of the C.S. Peirce Bicentennial International Congress*, Lubbock: Texas Tech University Press, 1981, 155-168) have duly insisted in the geometrical background (Hamilton, Cayley, Clifford, Klein, Listing, Riemann, Bolzano, Grassmann) of many fundamental peircean ideas. Nevertheless, we are still in need of a detailed study which may connect Peirce's early geometrical representations of the logic of relatives with his later topological insights (peircean *continuum*, continuity logic, existential graphs).

[66] A. Johanson's "protocompactness" in a modern pointless continuum can be seen in fact as a property which would insure the adequate glueing of coherent *possibilia*. See A. Johanson, "Modern Topology and Peirce's Theory of the Continuum", *Transactions of the Charles S. Peirce Society* XXXVII (2001), 1-12.

[67] May be a "generic cobordism" could be that "unnoticed condition in the general hypothesis of a collection which requires this mergency of individuals", unknown condition that Peirce considered the key of the "paradox" of the *continuum*. See "Multitude and Continuity" [c.1897; NEM 3.100].

Chapter II.
Some XX^{th} Century Mathematical Perspectives

Continuing an investigation of Peirce's continuum, we show how, following several independent paths, XX^{th} century mathematical logic has rediscovered (and amplified, to a better understanding) many aspects related to the genericity, reflexivity and modality of Peirce's continuum. From Peirce's wide legacy of ideas around the continuum, two trends have caught particular attention: his vindication of infinitesimals and his relations reduction thesis. Since thorough works on these subjects have appeared[68], we will not discuss them further here, and we will concentrate in other areas related to Peirce's continuum less well deserved. In particular, we claim that an understanding of modern methods in topological model theory and in category theory are extremely useful to disentangle the riddle of Peirce's continuum.

II.1. The primordial continuum

In the same years in which Peirce and Cantor wrote on the continuum, Giuseppe Veronese presented an alternative vision of the continuum, close to Peirce's in many respects. Veronese considers a "whole intuitive continuum" on which webs of points are just reference systems, and cannot fully capture the underlying continuous "fundamental form" –a feature shared by the reflexivity (and thus the inextensibility) of Peirce's continuum–:

> The rectilinear intuitive continuum does not depend on the system of points which we may think on it. Never a system of points can give, in an absolute sense, the whole intuitive continuum, since a point has no parts. [...] We shall see in our geometrical considerations that a system of points can represent the continuum sufficiently, and can do nothing more. The rectilinear continuum is not composed of points, but of sections (*tratti*), each of which joins two distinct points and is itself continuous. [...] *Introduction*: A fundamental form is a one-dimensional system which is homogeneous, i.e. identical in the position of its parts. [...] *Hypothesis VII* (homogeneity of the fundamental form). Every segment, where the ends vary in opposite directions and which become unlimitedly small, contains an element outside the domains of variability of its ends.[69]

The coincidence with Peirce, both in concepts and language, is deep. Veronese, a first rate italian mathematician, would then provide an extensive technical development of his "fundamental form", a task that Peirce's more limited skills in modern mathematical analysis could not undertake. The homogeneity of the fundamental form –extending the domain of variation of the continuum and guaranteeing enough infinitesimals ("elements outside" boundaries of sections "unlimitedly small")– corresponds, in Veronese, to the homogeneity of the *possibilia* realm in Peirce's continuum, insuring supermultitudinous "monads" around each point, or actual break, on the continuum. Veronese's continuum –intuitive, prelogical, pretopological– starts from a *non set-theoretic* notion of emptiness, a weaving and amalgamating synthetic notion which can be viewed as a smooth fluid, both finite and unlimited, in which parts melt naturally with the whole[70]. Veronese's continuum, as well as Peirce's, is *non archimedean*, since the archimedean property of the cantorian continuum[71] is just a way to force the continuum to be captured by standard natural number scales. Beyond Cantor's model, beyond analytic number representability, over a generic synthetic ground –"smooth" or "plastic"–, lie Veronese's and Peirce's continua.

In the first stage of Brouwer's thought (1907-10), the continuum appears as a primordial synthetic intuition. Brouwer draws images akin to Peirce's and Veronese's:

> The continuum as a whole is intuitively given; a construction of the continuum, an act which would create by means of the mathematical intuition "all its points" is inconceivable and impossible.[72]

Brouwer starts from a wholly generic continuum, on which acts and reacts the "primordial intuition" of mathematics, the "auto-conscious" possibility of human mind, able both to *mark* the continuum and to *observe* the mark, producing thus the brouwerian "two-oneness" which allows to develop intuitionistic number theory (integers, constructible reals):

> ...the intuition of two-oneness, the primordial intuition of mathematics which immediately creates not only the numbers one and two but all finite ordinal numbers...[73]

> In the Primordial Intuition of two-oneness the intuitions of continuous and discrete meet: "first" and "second" are held together, and in this *holding-together* consists the intuition of the continuous (continere = hold together).[74]

> In the temporal two-ity emerging from time-awareness one of the elements can again and in the same way fall apart, leading to temporal "three-ity", or three-element time sequence is born. Proceeding this process, a self-unfolding of the primordial happening of the intellect, creates the temporal sequence of arbitrary multiplicity.[75]

In Brouwer's second period (1917-30), the Dutch mathematician articulates again from scratch his vision of the continuum, generating it constructively with his choice sequences; then, the intuitionistic continuum develops dynamically and becomes a *variable set*. A *reflexive* constructive process seems to happen here, similar to the ones we signaled in Peirce and Veronese: from a global intuition of the continuum one goes over to local constructions, which try to reflect the original view –"one", "primordial", "fundamental"[76]–.

In the intuitionistic continuum several existential proof arguments (valid in Cantor's model) do not hold, and the law of excluded middle fails (as in Peirce's continuum). Also, Brouwer's and Peirce's continua are both "supermultitudinous", in the sense that both possess the highest consistent cardinality for a mathematical concept inside their respective theoretical contexts. Indeed, according to Peirce, the size of the continuum is the limit of denumerable iterations of the exponential (maximum size in Peirce's system, since he didn't allow the use of arbitrarily high ordinals), and, according to Brouwer, the size of the continuum is the biggest between only four recognizable constructive sizes (finite, denumerable, non-denumerable, primordial continuity). Of course, these versions of the continuum cannot be implemented in classical set-theory, where arbitrary ordinals do exist and where Cantor's theorem holds ($||\wp(X)|| > ||X||$).

The generic intuition of the continuum will not be lost throughout the XX^th century and will be retrieved with force by the Field medallist, René Thom:

> Here I would like to face a myth deeply anchored in contemporary mathematics, namely that the continuum is engendered (or defined) as arithmetic unfolds through the sequence of natural numbers. [...] I estimate, on the contrary, that the archetypical continuum is a space which possesses a perfect qualitative homogeneity; I would like to say that two "points" are always equivalent by means of a continuous sliding (eventually local) of the space on itself; unfortunately the very notion of a "point" already presupposes a break of spatial homogeneity. [...] The notion of place (Aristotle's τοπόζ) could perhaps help to access a rigorous definition, since places can serve as an open basis for a topology. A decreasing sequence of nested intervals could converge to that minimal element: a point. Our archetypical continuum possesses no structure by himself (metrical or simply differential): the only demanded property is its qualitative homogeneity.[77]

Many deep similarities draw near Thom's postulates and Peirce's indications on the continuum: Thom's "perfect qualitative homogeneity" corresponds to Peirce's perfect generality, Thom's critic on the "myth" of the arithmetization of the continuum recalls Peirce's assertion that number cannot possibly capture continuity, Thom's understanding of points as homogeneity breaks matches Peirce's vision of points as continuity breaks, Thom's attempt to restore the notion of "place" as a basis for a rigorous definition of the continuum corresponds to Peirce's intent to construct a neighbourhood logic connatural with the generic sliding of the fluid in a drop of ink, Thom's fundamental insistence that the "archetypical continuum" must really be an archetype without additional structure corresponds to Peirce's basic insight that the continuum must be a purely relational General ("free" in the sense of mathematical category theory, as we shall soon see).

Thom explains the passage from the continuum to the discrete by means of "cuts" (breaks, discontinuities), local marks which could serve as actuality spots for the brouwerian two-oneness:

> We can say nothing of that perfect continuum but it is an *unutterable mystique*: it carries no mark, no point, does not admit any orientation, nothing can there be identified. How, then, happens in this medium the first intrusion of the discrete? [...] *The intrusion of the discrete in the continuum manifests itself by the cut*. [...] On the line, the point appears as a cut: it helps to hinder the left half-line (*Dg*) from the right half-line (*Dd*). [...] Here intervenes a vision proper to Aristotle. As a point *O* is marked on the line (*D*), the line divides in two potential (δυναμει) half-lines; but to get separation in act (εντελεχεια), one needs that the point *O* unfold in two points, *Og* left adherent to *Dg*, *Od* right adherent to *Dd*, and only then the two half-lines closed in *O* reach the existence in act as two separate entities. The points *Og*, *Od*, although different (boundaries of different entities) are nevertheless together (αμα), and we have passed, in *O*, from a continuity situation to a contiguity situation; thus the celebrated formula: *entelechy severs*.[78]

Cut intrusions in the continuum, and its *boundaries unfolding*, allow thus a natural flow from continuity to contiguity, a process that Peirce had already signaled as basic in the progressive determination of the general. A general theory of boundaries should be of great profit, then, in order to obtain a better understanding of Peirce's continuum. Thom's general theory of *cobordism* is in fact such a theory, but its range of application is yet restricted to differential geometry. A natural path to follow –hard but important– would consist to "free" axiomatically cobordism theory from its differential structure and to abstract it towards the general. Even if such a path seems still remote, we believe one of the natural apparatus to clear the route is now in place: the very powerful abstract categories of relations ("allegories") of Peter Freyd.

Freyd's allegories[79] provide precisely a general methodology which allow to pass from a structured class to its "free" skeleton. Following an ubiquitous procedure in categorical logic, Freyd shows that, departing from pure type theories with certain structural properties (regularity, coherence, first order, higher order), one can construct *in a uniform way* –through an architectonic hierarchy completely controlled– *free* categories which reflect the given structural properties (regular categories, pre-logoi, logoi, topoi).[80] Lean, free categories are then able to be reflected in *any* other category with similar properties: Freyd achieves thus the amazing discovery of *initial archetypes* in mathematical theorization.

It is not therefore too risky to conjecture success for what we would like to call the *allegorical program for the continuum*: to construct a hierarchy of partial, converging, models for Peirce's and Thom's archetypical continuum, starting from appropriate differential structures and "freeing" them towards genericity, through Freyd's allegorical representation machinery. It should also be noticed that the natural context of allegories can provide several other benefits in an assessment of Peirce's continuum. For example, on one side, Peirce's generic triadic relation, which he signals as an eventual key to the continuum, fits perfectly in the axiomatic framework of allegories, where the connecting modes of abstract relations are studied in full generality. On another side, Peirce's antithesis between generality and vagueness, found by Lawvere to be a full categorical adjunction between the universal and existential quantifiers (∀ -- ∃), gets an immense algebraic richness through the allegorical machinery. On yet another perspective, a wide range of partial modalities hidden in Freyd's calculi could be used to model Peirce's flow of modalities over the continuum.

II.2. The large set-theoretic continuum

Always independently of Peirce's original ideas on the continuum, mostly unknown to the scientific community, other generic aspects of Peirce's continuum have been modeled through other mathematical constructions of XXth century mathematics. In particular, the generic size of Peirce's continuum, its "supermultitudeness", its multitude (cardinal) larger than all other multitudes, has revived in the *super-infinity* of some collections in alternative set theories: the super-infinity of the class of Conway's "surreal" numbers in NBG (cantorian) set theory, and the super-infinity of the set of natural numbers in Vopenka's (non-cantorian) AST.

The class *No* of Conway's surreals[81] can be axiomatized (Ehrlich), in NBG set theory[82], by means of axioms for *non archimedean* ordered fields together with an axiom of *absolutely homogeneous universality*:

> A model *A* for a theory *T* in a language *L* will be said *absolutely homogeneous universal* [B. Jónsson, 1960] if and only if it is *absolutely universal* with respect to *T* (i.e., every model of *T* in *L* can be embedded in *A*) and it is *absolutely homogeneous* with respect to

T (i.e., given any two substructures of *A* that are models of *T* in *L*, whose universe are sets, and an isomorphism between them, the isomorphism can be extended to an automorphism of *A*.[83]

Besides the very interesting fact that *No* can be axiomatized through generic properties of homogeneity and universality –following closely the (independent) general guidelines of Peirce's continuum– it is fundamental for us another theorem which guarantees the supermultitudeness of *No*, since *No* contains the class *On* of *all* ordinals[84] in NBG. Thus, Conway's *No* finely models several aspects of the genericity of Peirce's continuum, even if it lies far away from reflecting its other reflexive and modal properties. In fact –as well as with all other contemporary mathematical constructions we are aware of– just some *partial* properties of Peirce's continuum seem able to be reflected in a given mathematical model (confirming thus, inductively, what the pragmatic maxim would anyway foresee).

In sharp contrast to NBG, Vopenka's AST[85] can be truly considered an alternative set theory. Vopenka distinguishes, as in NBG, classes and sets, but he introduces above all a really strong *asymmetrization*, which breaks the usual set theoretic equivalence between intensionality and extensionality. In AST, Zermelo's separation axiom can fail: not every subclass of a (AST-)set has to be a (AST-)set. The asymmetrization signals, on one hand, that not every intensional property has to yield an extensional set, and, on the other hand, that very little of the indefinite and infinite range of intensional *possibilia* can be effectively actualized –a conception Vopenka founds in Bolzano[86] and, as we saw before, Peirce independently took up again–. As a consequence of the radically new *axiomatic contextualization* of Vopenka, $\wp(X)$ no longer can be actualized when *X* is infinite (even if, for finite sets, ZF and AST do agree). The theory possesses, then, *just two* infinite cardinalities: the one of the class of usual natural numbers (*N*), and the one of the class *An* of "finite" natural numbers (defined as those for which every subclass is in fact a set). In this theory, far away from what happens in ZF, *N* plays part of the role of the continuum, being a super-infinite class.

From another set-theoretic perspective, even if inside ZF no global supermultitudinous or reflexive models can be found, in the two better known ZF extension scales –the great cardinals scale and the forcing axioms scale– diverse local properties can be found, which reflect one scale into the other and which may correspond to fragments of genericity and reflexivity in Peirce's sense. Going farther than bounded genericity in forcing[87] –which builds up over *particular* classes of orderings (ccc, proper, etc.) in order to insure nice extension properties of the associated models (cardinal preservations, iterations, etc.)– and *abstracting away* from those local orderings, it would be extremely important to axiomatize a *generic notion of genericity*. In such an endeavour a welcome union of set-theoretic and category-theoretic tools would have to take place, and Peirce's continuum could turn to be approached as a maximal generic extension of Cantor's "first embryo" of continuity.

The reflexivity of Peirce's continuum, and, therefore, its inextensibility, are constitutive characteristics that do not seem possible to be modeled naturally inside ZF (where points *do* build up sets). Linked with this obstruction lies the *intensionality* of Peirce's continuum (similar to the "primitive" intensional versions of Veronese's, Brouwer's or Thom's continua), to which are superposed afterwards extensional local fragments ("number scales") in order to gain relative control. Thus, it may be relevant –as Vopenka advocates going back to Bolzano's intensional domains– to try to develop versions of the continuum in axiomatic settings where Frege's abstraction principle becomes asymmetrically weakened[88]. From the perspective of required axioms to capture an intensional, inextensible and generic continuum such as Peirce's, Zermelo's local separation axiom may be still too stringent. A further local asymmetrization of the principle (favoring the rise of intensional concepts, as in existential graphs) could be in order. In Peirce's cosmological continuum, the realm of *possibilia* and the intensionality of real potentials reign over the actual extensionality of existence; similarly, in his logical continuum (*continuously* with the cosmological continuum!), a clear primacy of the intensional should be reflected in local axiomatic settings.

Pre-eminence of intensionality would convey an important support to the inextensibility of Peirce's continuum. Indeed, an asymmetrization of Zermelo's separation axiom immediately gets rid of the *a priori* existence of points: since only some formulas produce associated classes, the singletons $\{a\}$ not always need to exist (they are associated to formulas $x=a$, which could turn to be not available in the theory if the parameter a is not constructible). Also, perhaps with some sort of local paraconsistent logic, some manipulations of contradictory intensional domains could be developed –in the potential realm– without yet facing the associated contradictory extensional classes –in the actual realm– which would trivialize the system, thus confering a greater flexibility to a generic approach ("free" of actual bonds) to the continuum. It should also be noticed that brilliant mathematicians, such as Jean Bénabou[89] and Edward Nelson[90] (as well as Thom) consider that the intension-extension symmetry, creed of contemporary mathematics, must be broken.

II.3. The category-theoretic continuum

Constructed as a generic environment for the transversal study of information transfers between mathematical structures –a weaved environment where diverse synthetic "universal properties" are contrasted, an intensional environment where extensional objects are not *a priori* needed–, the mathematical theory of categories is the environment of contemporary mathematics which better can be fused with Peirce's thought, and where perhaps the greater number of tools and models can be found to faithfully approach both Peirce's general architectonics and Peirce's particular ideas.[91] The continuum –vessel and bridge between the general and the particular– is therefore specially well suited to be understood categorically. The paradigm of the mathematical theory of categories[92] –"arrows, *no* elements"; synthesis, *no* analysis; relational, contextual, external knowledge, *no* monolithic, isolated, internal knowledge– reflects nicely Peirce's pragmatic maxim.[93] In category theory the pragmatic dimension becomes evident through diverse functorial readings ("interpretations") between "concrete categories". As invariants of a generic functorial back-and-forth emerge –solidly: *theorematically*– "real" universal notions, definable in any "abstract category", beyond its eventual existence (or non-existence) in given particular categories. Category theory provides thus the more sophisticated technical arsenal, available in the present state of our culture, which can be used to prove that *there do exist real universals*, vindicating forcefully the validity of Peirce's scholastic realism.

One of the fundamental *visions* that category theory supplies is compactly codified in Yoneda's lemma, a mathematical result of utmost simplicity but which explains in a deep way the *generic presence* of the continuum in any consideration of reality. Yoneda's lemma shows that *any* "small" category can be faithfully embedded in a category of "presheaves" (functors to sets), where "ideal" (or "non standard") objects crop up to *complete* the universe, turning it continuous:

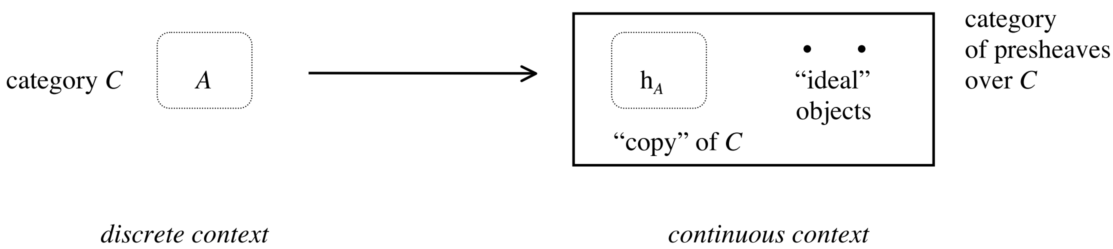

Figure 9.
Yoneda's lemma: generic presence of the continuum

Diverse forms of continuity are hidden behind Yoneda's lemma. "Representable" functors h_A symbolize (in Peirce's sense) all interrelations of A with its context and *preserve limits*: they are "continuous". The presheaf category where the initial category is embedded is a "complete" category, in the sense that it possesses *all categorical limits*: it is therefore a natural continuous environment. Even deeper, Yoneda's lemma is the natural tool to describe the "classifier objects" (semantic codifiers) in presheaf categories: the truth notions turn out then to be –in an *natural* way– pragmatic notions, weaved with the continuum where they lie. The emergence of "ideal" objects as the "real" is tried to be captured –explicit and unavoidable in Yoneda's lemma, penetrating and permanent in any form of mathematical creativity– agrees with the peculiar mixture of realism and idealism present in Peirce's philosophy. The continuous bottom emerging in Yoneda's lemma is yet another indication that Peirce's global synechism can count on amazing local reflections to support its likeliness.

Some presheaf categories serve in turn as appropriate places for the construction of *internal* models of the continuum, where some aspects of the genericity and inextensibility of Peirce's continuum become actualized. One of those environments where a "synthetic geometry of the continuum" can be produced is the presheaf category $C = Set^{L^{op}}$ where L is the category of formal C^∞ varieties (Lawvere, Reyes, Moerdijk)[94]. A "copy" in *C* (through Yoneda) of the cantorian real line, called the "smooth line", acquires very nice properties –by virtue of its *new relationships* with the enhanced presheaf environment– which accommodate some requirements of Peirce's continuum: the smooth line is non archimedean, possesses infinitesimals, can not be determined by points, contains a generic (non standard) copy of the naturals. In another truly surprising technical way, but perfectly in tune with peircean semiotics, this shows how a "copy" (more precisely: an interpretant) of an incomplete concept, in a given context, can complete itself naturally in another, richer, context.

II.4. The sheaf continuum

On another hand, other internal models in *sheaf* categories[95] can detach (prescind, make a "prescision" in Peirce's terms) certain properties *fused* together in the cantorian real line (*R*), showing again that *R* contains too much *spare* structure and that it is not generic enough (recall Thom's advocation that the archetypical continuum should possess "no structure" beyond its "qualitative homogeneity"). Indeed, in any sheaf category *Sh(O(T))* over a topological space (*T*,*O(T)*), one can construct (Troelstra, van Dalen) diverse copies of the cantorian real line[96]. In the specific case of the category *Sh(O(R))*, the copies are different according the construction is done through Dedekind cuts (R^d) or through Cauchy sequences (R^c), yielding closure properties neatly detached from an intuitionistic perspective (R^c is real-closed, R^d is not). Even if, again, current intuitionistic sheaf models do not seem more than "first embryos" of continuity, the *logic of sheaves* underlying those models –which technically provides a finer handling of genericity and neighbourhood logic– should be of great help in an appropriate global axiomatization of Peirce's continuum.

Sheaf logic, proposed in a very ductile and fruitful form by Xavier Caicedo[97], includes a wide range of intermediate logics between intuitionistic logic and classical logic. Given a topological space, Caicedo defines a natural local forcing on open sets, and he uses it (with all rigour of modern mathematical logic and, once again, independently of Peirce) to carefully emphasize Peirce's fundamental idea that truth is generically *local* and not just punctual[98]: something is valid in a point if and only if it is valid in a neighbourhood around the point. Sheaf logic *coheres* accurately a lot of Peirce's detached ideas ("*detached ideas* on vitally important topics"). Caicedo's results handle well the problematics around genericity and neighbourhood logic (recall the "double sigma" which codes Peirce's continuum – first chapter) and open fascinating new perspectives. The construction of a theory of *generic models* allows to obtain –in a uniform way, as simple corollarial structures in appropriate sheaves– the fundamental theorems of classical model theory (completeness, compactness, types omission, Los' theorem for ultraproducts, set theoretic forcings), while the study of interconnections between usual punctual semantics (*à la* Tarski) and local sheaf semantics allows to reconstruct classical truth, in the sheaf fibers, as natural *limit* of intuitionistic truth, characteristic of its global sections.[99]

Caicedo's contributions show that –as newtonian mechanics can be seen as a limit in Einstein's relativity, or euclidean space can be seen as a limit in Riemann's geometry– classical logic deserves to be understood as a limit in sheaf logic. The awareness of this bordering situation can be interpreted in two complementary ways "vitally important". On one hand, it explains (in a precise conceptual and technical way, not just dogmatic) the pre-eminence of classical logic in the development of XX[th] century mathematics[100], since classical logic turns out to be the natural logic which better fits the “cantorian program” –construction of mathematics as punctual sum of ideal actualizations, in an static and Platonic context–. On the other hand, it opens huge perspectives on the *continuum* of intermediate layers between intuitionistic and classical logic, and locates sheaf logic as the natural logic which better seems to suit what we would like to call a *peircean program for mathematics* –construction of mathematics as relative web of real possibilities, in an evolving and Aristotelean context–.

Other findings of Caicedo[101] –on global continuous operations which codify structural properties of extensions of first order classical logic– yield an illuminating perspective on Peirce’s fundamental weaving between generality, continuity and relative logic. Applying topological methods in model theory, Caicedo shows that general axioms in abstract logics coincide precisely with continuity requirements on algebraic operations between model spaces, and he establishes an extensive list of correspondences between topological and logical properties, many of them based in the discovery that *uniform continuity* of natural operations between structures hide strong logical contents. Caicedo’s theorems can be interpreted in various ways to elucidate Peirce’s “cryptic” motto: • *continuity* = *genericity* via *relative logic* • On one side, following a straight global reading of Caicedo’s results, we can see that the “general” (axioms of abstract model theory), filtered through the web of relative logic (first order classical logic), yields a natural continuum (uniform topological space by way of “local” elementary equivalence[102]; uniform continuity of logical operations in that web: projections, expansions, restrictions, products, quotients, exponentials[103]). On another side, for example, following a more detailed reading, the fact that closure under relativizations in an abstract logic is *equivalent* to comparing adequate uniform topologies in model spaces[104], thus demarcating and detaching many logical transfers, shows that the “relative” and the “continuous” can coincide in a level of utmost abstraction, “free” and “general”.

As we have seen, multiple advances in XX[th] century mathematics –alternative set theories, category theory, sheaf logic, topological logic– help to determine more accurately Peirce’s ideas on the continuum, with regard to global *genericity and* *reflexivity* and their local counterparts (generic relations, vagueness, neighbourhood logic). In spite of those achievements, lesser can be found to model in a correct way Peirce's continuum as a "replenished" modal realm, where all universe of *possibilia* could fit. A path to be explored is Jan Krajicek's modal set theory (MST)[105], where one can work with an *irrestrictive* abstraction principle, but where certain constructions have to be modalized in order to avoid the inconsistency of the theory[106]. In the MST context, a natural problem would be to define (with perhaps additional axioms) a supermultitudinous continuum and to show its relative consistency; such a definition seems plausible since the abstraction principle can be dealt in all its global potentiality, beyond actual multitudes. Krajicek's theory is constructed over a classical basis: first order classical logic plus modal calculus T. Nevertheless, as we have signaled, intuitionistic logic –more akin to variable sets and topologies, closer to a full treatment of the continuum– could be the basis of a similar system, constructed in a more specific way to apprehend Peirce's continuum. In this sense, another natural problem could be to propose an intuitionistic modal set theory (beginning with a variation MSTI) and to explore definitions and connections, in the new theory, of the intermediate concepts which approach softly the continuum (particularly, sheaves and logico-topological methods).

Besides Krajicek's MST, another path would have to be followed if we are looking for modern tools to understand (and develop) Peirce's modal continuum: Gonzalo Reyes' very interesting work on *bi-Heyting algebras* (Heyting algebras[107] with a "difference" co-dual to Heyting's implication). In these algebras, several pioneering Lawvere's insights on abstract *boundary* operators can be nicely formalized, and it can be shown that many modal operators turn out to be *limits* of natural iterations of the difference and the negation operators available in the bi-algebra[108]. The classifier object in *any* presheaf topos possesses a bi-Heyting algebra structure and, thus, any presheaf topos counts with an infinite hierarchy of intermediate modalities. In this way, the presence of continuous modalities turns out to be much more ubiquitous than expected, pointing again to the immense richness lying in a multifarious category-theoretic approach to Peirce's continuum.

Beyond the diverse *partial* tools yielded by contemporary mathematics to approach Peirce's continuum, remains the deep problem of *unifying* those partial models in a coherent global context, *in case such an unification is possible*. It is not clear, indeed, if there exist intrinsic limits to a global understanding of the continuum, and –even if we renounce to find a "monster" model which encompasses at the same time genericity, reflexivity and modality– if it is possible to find a "free" pragmatic theory which gradually could weave the continuum. As an objective for coming work –abduction to be contrasted inductively by future deductions– we conjecture that such an "skeleton" theory *should* in fact be possible to be constructed, in terms of mathematical category theory –following the "allegorical program for the continuum"– and that the indeterminate universality of the continuum should be able to incarnate progressively in concrete categories, laying local differential marks that should nevertheless be able to be *reintegrated* functorially –completing the "peircean program for the continuum"–.

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[68] On Peirce's infinitesimals (and, particularly, on *nilpotent* infinitesimals, closer to Peirce's ideas) see T. Herron, "C.S. Peirce's Theories of Infinitesimals", *Transactions of the Charles S. Peirce Society* XXXIII (1997), 590-645. On Peirce's reduction thesis see R. Burch, *A Peircean Reduction Thesis. The Foundations of Topological Logic*, Lubbock: Texas Tech University Press, 1991. Burch's deep work is, strangely, still considered open to controversy, but in fact Burch proves in a definitive way that ternary relations *cannot* be reduced to binary and unary relations *over a language of continuous operators (junctions) on relations*, capturing thus Peirce's *topological* logic. The fact that ternary relations can be reduced to binary and unary relations over a language of discrete operations on relations (the usual reduction of relations to sets of couples) does not hinder in any way Burch's results.

[69] Giuseppe Veronese, *Fondamenti di Geometria* (1891) (§55, footnote), cited by Detlef Laugwitz, "Leibniz' Principle and Omega Calculus", in: J.M. Salanskis, H. Sinaceur (eds.), *Le Labyrinthe du Continu*, Paris: Springer-Verlag, 1992, p.154.

[70] For this description we have used R. Peiffer-Reuter's, "Le Fond Lisse et la Figure Fractale: l'Idée du Continu chez Natorp et Veronese", in: Salanskis-Sinaceur (op.cit.), p.98. According to Peiffer-Reuter, Veronese's intuitive continuum is then mathematicized by an "avalanche" of scales, both in the infinitely small and the infinitely large, constructing thus a local and partial *reflection* of the underlying global and generic "fond lisse". See also, Paola Cantù, *Giuseppe Veronese e i Fondamenti della Geometria*, Milano: Unicopli, 1999, particularly chapter 2, "Il Continuo non Archimedeo", pp.87-164.

[71] Archimedean axiom: given any pair of positive reals, any of them can be exceeded by an integer multiple of the other.

[72] L.E.J. Brouwer, "On the foundations of mathematics" (1907; doctoral thesis), cited in: W. P. van Stigt, *Brouwer's Intuitionism*, Amsterdam: North-Holland, 1990, p.323.

[73] L.E.J. Brouwer, "Intuition and Formalism" (1912), *ibid*, p.149.

[74] L.E.J. Brouwer, "Die mögliche Mächtigkeiten" (1908), *ibid*, p.155.

[75] L.E.J. Brouwer, “Willen, Weten, Spreken” (1933), *ibid*.
[76] G. Locke, “Peirce’s Metaphysics: Evolution, Synechism, and the Mathematical Conception of the Continuum”, *Transactions of the Charles S. Peirce Society* XXXVI (2000), 133-147, also calls “aboriginal” the “primordial” continuum.
[77] R. Thom, “L’Anteriorité Ontologique du Continu sur le Discret” (1992), in Salanskis-Sinaceur, op.cit., p.141.
[78] *Ibid*, p.142. Thom’s remarks could help to understand better the enormous philosophical significance buried in Peirce’s existential graphs cuts. In fact, Peirce’s alpha and gamma ovals produce actual, discrete, formulas over the underlying continuum of the sheet of assertion. In a forceful way, “entelechy severs” meanwhile the graphs are illatively transformed.
[79] P. Freyd (with A. Scedrov), *Categories, Allegories*, Amsterdam: North-Holland, 1990. The work lasted almost twenty years (1972-90) in the making. Without doubt, it should be considered as one of the highest points of XXth century mathematics, height nevertheless still not valued enough by the mathematical community.
[80] Procedure T (theory) $\rightarrow$ A_T (allegory) $\rightarrow$ MapSplitCor(A_T) (category), which produces a “free” result when one starts from a “pure” type theory, and which shows in every step (relationality, identity merging, partial inverses, functionality) how a general mathematical conglomerate is being “filtered” towards the general. *Ibid*, p.277.
[81] J.H. Conway, *On Numbers and Games*, London: Academic Press, 1976.
[82] “NBG” for von Neumann - Bernays - Gödel. The theory distinguishes (arbitrary) classes and sets (classes which are member of other classes), allowing simpler infinity handlings than ZF. Nevertheless, NBG’s and ZF’s construction schemes are very similar and their proof power is identical (equiconsistent theories).
[83] P. Ehrlich, “Universally Extending Arithmetic Continua”, in: Salanskis-Sinaceur (op.cit), p.169.
[84] It should also be noted that E. Nelson’s model of Robinson’s non-standard analysis is another candidate for supermultitudeness, since it provides “plenty of natural-looking subsets (...) which have more points than any cardinal number in Zermelo-Fraenkel set theory” (see T. Herron, op.cit., p. 620). A further supermultitudinous model for the continuum, adjoining arbitrary ordinal lengths, can be found in W.C. Myrvold, “Peirce on Cantor’s Paradox and the Continuum”, *Transactions of the Charles S. Peirce Society* XXXI (1995), 508-541.
[85] P. Vopenka, *Mathematics in the Alternative Set Theory (AST)*, Leipzig, 1979. Also: A. Sochor, “The alternative set theory”, in: *Set Theory and Hierarchy Theory*, New York: Springer, 1976, pp. 259-273.
[86] See J. Sebestik, *Logique et Mathématique chez Bolzano*, Paris: Vrin, 1992, pp. 471-472. For Vopenka, the domain of actual sets is no more than a “minute island of actuality in the ocean of potentialities” (*ibid*).
[87] Woodin’s brilliant work, which finds *natural* axioms at level $H(\omega_2)$ to decide the continuum hypothesis (in the sense that $2^{\aleph_0} = \aleph_2$), can be seen as a protended effort to define a universal homogeneous order for $H(\omega_2)$ and to guarantee enough generics for that order. The *bounded* homogeneity and genericity thereby studied should be transcended beyond ω_2. For references, see the (compact and straightforward) mimeo: H. Woodin, *The Continuum Hypothesis*, University of California (in particular, p. 41), or also his (gigantic and cumbersome) monograph (934 pp.): H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, Berlin: De Gruyter, 1999.
[88] Frege’s abstraction principle puts on the same level intension and extension: (FAP) for all “intension” (formula $\varphi(x)$) there exists a corresponding “extension” (class $\{x: \varphi(x)\}$), and vice versa. The *global* equivalence demanded by (FAP) leads immediately to Russell’s contradiction (considering the formula $\varphi(x) \equiv x \notin x$). Zermelo’s separation (or comprehension) axiom (basis of the system ZF, removing all known contradictions) breaks the global symmetry intension-extension, but retains a *local* equivalence between them: for all $\varphi(x)$ and *for all A* there exists a class $\{x \in A: \varphi(x)\}$. Peirce’s continuum would seem to need a further break of symmetry at local levels.
[89] J. Bénabou, “Rapports entre le Fini et le Continu”, in: Salanskis-Sinaceur (op.cit.), p.178: “A certain number of signs [e.g., non-standard analysis, topos theory, according to Bénabou] show that the

essential assumption of «set-theoretic creed» –namely, that only concepts in which coincide extension and comprehension [e.g., intension] can be apprehended by mathematics– begins to falter. [...] The indispensable distinction, underlined by Thom, between extension and comprehension of a concept, impossible in set theory, begins to be reckoned in various ways, even if still by a large minority".

[90] Like Thom, Nelson criticizes certain mathematical "myths" and "beliefs" (such as ZF's consistency) which would rather seem religious. See E. Nelson, "Mathematical Mythologies", *ibid*, p.156.

[91] An excellent use of the mathematical theory of categories to pinpoint and extend Peirce's semiotics can be found in R. Marty, *L'algèbre des signes. Essai de sémiotique scientifique d'après Charles Sanders Peirce*, Amsterdam: Benjamins, 1990. Nevertheless, beyond Marty's deep work, we don't know any other sustained effort to apply the mathematical theory of categories to Peirce's thought. Several indications of such a program have been here only recorded, but we hope to develop them at length in the future.

[92] One should not confuse the mathematical theory of categories and Peirce's cenopythagorean categories (One-Two-Three): even if they overlap perfectly in complementary levels and readings, the two theories cover completely different methods and objectives.

[93] For a *diagrammatic* presentation of the maxim, much in the vein of the mathematical theory of categories, see our next chapter.

[94] I. Moerdijk, G. Reyes, *Models for Smooth Infinitesimal Analysis*, New York: Springer, 1991.

[95] Informally, a "sheaf" is a "presheaf" which can glue, through generic elements, the diverse *compatible* information collections codified in the presheaf.

[96] A.S. Troelstra, D. van Dalen, *Constructivism in Mathematics*, Amsterdam: North-Holland, 1988.

[97] X. Caicedo, "Lógica de los haces de estructuras", Revista de la Academia Colombiana de Ciencias, XIX (1995): 569-586.

[98] Peirce's "ink spot" and his logical analysis of the boundary are repeated (independently) by Caicedo in an almost identical form as Peirce does. *Ibid*, p.570, figure 1.

[99] In his sheaf logic –constructed systematically in an intermediate layer between Kripke models and Grothendieck topoi, profiting both from concrete particular examples and abstract general concepts– Caicedo works in a crossroad of algebraical, geometrical, topological and logical techniques. The *back-and-forth* between the generic and the concrete, as well as his transversal crossing techniques, show that in his very *method* of research (beyond similar objectives) Caicedo stands very close to Peirce.

[100] Lindström theorems also explain carefully the natural pre-eminence of classical logic with respect to very specific properties (Löwenheim-Skolem) of Cantor's set theory, but they show in turn that classical logic is very *rigid* with respect to its basic structural properties (booleanness, relativization, compactness).

[101] X. Caicedo, "Continuous operations on spaces of structures", in: M. Krynicki, M. Mostowski, L.W. Szczerba (eds.), *Quantifiers: Logics, Models and Computation*, Dordrecht: Kluwer, 1995, vol. I, 263-296. Xavier Caicedo: "Compactness and normality in abstract logics", *Annals of Pure and Applied Logic* 59 (1993), 33-43.

[102] X. Caicedo, "Continuous operations on spaces of structures", op.cit., p.266.

[103] *Ibid*, p.273.

[104] *Ibid*, p.276.

[105] J. Krajicek, "Modal Set Theory", preprint, University of Prague, 1985.

[106] Global MST could turn out to be inconsistent. Krajicek only proves relative consistency of some of its fragments.

[107] Heyting algebras provide a canonical semantics for intuitionism, in the same way Boolean algebras codify classical semantics. Heyting algebras are closely related to topologies provided with simple set-theoretic operations: the continuum continuously continues to be hidden in unsuspected places!

[108] G. Reyes, H. Zolfaghari, "Bi-Heyting Algebras, Toposes and Modalities", preprint, Université de Montréal, 1991.

Chapter III.
Architectonics of Pragmaticism.

Peirce's pragmatic architectonics can be seen as a sophisticated system, with multiple information channels and nested control layers, constructed to understand the world *simultaneously* in its more abstract generality and its more concrete specificity. The architecture of the system, with its pervasive reflections and overlapping frames, recalls the gothic cathedral evoked by *À la Recherche du Temps Perdu*, but transcends even the work of man, trying to capture a general architectural design in the natural world, a reality independent of communities of inquirers. Peirce's architectonics provides a wide arsenal of *crossing* instruments to understand in part a complex reality, where –in a frontier crossed over by constantly iterated and deiterated information– merge the richness of external cosmos and the multiplicity of semiotic systems interior to cultural communities. It is not therefore surprising that Peirce's architectonics supposes a *continuum*, which weaves cosmos and humanity, which systematically studies the crossing and bordering processes characteristic of any semeiosis, and which supports the possibility of contrasting the back-and-forth breedings of the edifice.

In the first part of this chapter we stress five basic structural spans (pragmatic maxim, general categories, universal semeiotics, determination-indetermination duality, triadic classification of sciences) which support Peirce's architectonics. We have emphasized a diagrammatic presentation of some of those arches, paying particular attention to a fully modalized diagram of the pragmatic maxim, which will be central to our latter concerns around a "local proof of pragmaticism". Then, in the second part of the chapter, we show how *explicit* continuity assumptions are strongly related to the steadiness of those spans.

III.1. Five Arches of Peirce's Architectonics

The pragmatic (then pragmaticist) maxim appears formulated several times throughout Peirce's intellectual development. The better known statement is from 1878, but more precise expressions appear (among others) in 1903 and 1905:

> Consider what effects which might conceivably have practical bearings we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object.[109]
>
> Pragmatism is the principle that every theoretical judgement expressible in a sentence in the indicative mood is a confused form of thought whose only meaning, if it has any, lies in its tendency to enforce a corresponding practical maxim expressible as a conditional sentence having its apodosis in the imperative mood.[110]
>
> The entire intellectual purport of any symbol consists in the total of all general modes of rational conduct which, conditionally upon all the possible different circumstances, would ensue upon the acceptance of the symbol.[111]

The pragmaticist maxim signals that knowledge, seen as a semiotic-logical process, is pre-eminently contextual (*versus* absolute), relational (*versus* substantial), modal (*versus* determined), synthetic (*versus* analytic). The maxim serves as a sophisticated *sheaf of filters* to decant reality. According to Peirce's thought, we can only know through signs, and, according to the maxim, we can only know those signs through diverse correlations of its conceivable effects in interpretation contexts. The pragmatic maxim "filters" the world by means of three complex webs which can "differentiate" the one into the many, and, conversely, can "integrate" the many into the one: a *representational* web, a *relational* web, a *modal* web. Even if the XX^th^

century has clearly retrieved the importance of representations and has emphasized (since cubism, for example) a privileged role for interpretations, both the relational and the modal web seem to have been much less understood (or made good use) through the century.

For Peirce, the understanding of an arbitrary *actual* sign is obtained contrasting all *necessary* reactions between the interpretations (sub-determinations) of the sign, going over all *possible* interpretative contexts. The pragmatic dimension emphasizes the *correlation* of all possible contexts: even if the maxim detects the fundamental importance of local interpretations, it also urges the reconstruction of global approaches, by means of appropriate *relational and modal glueings* of localities. A *diagrammatic scheme* of the pragmaticist maxim –which follows closely the 1903 and 1905 enunciations above stated– can be the following:

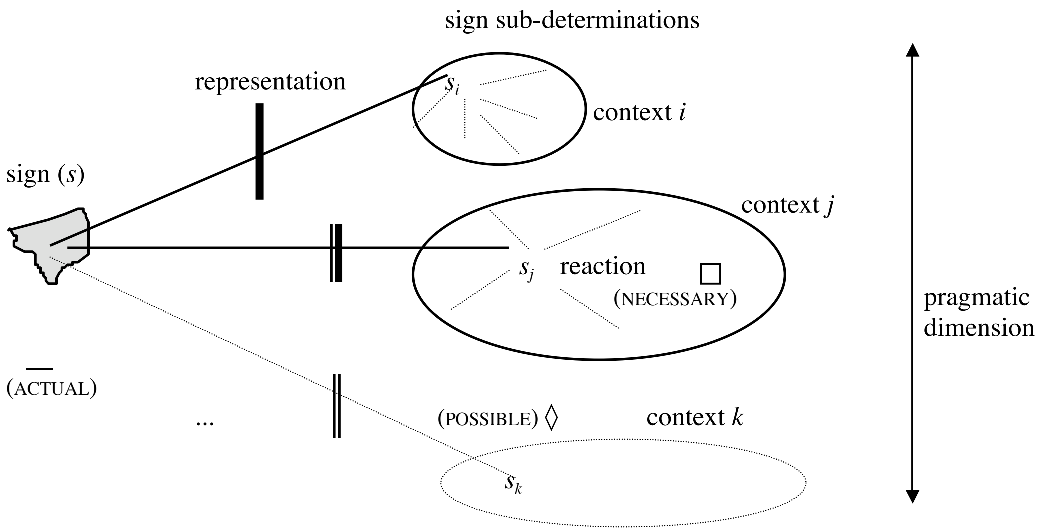


Figure 10.
Peirce's pragmaticist maxim

In our next chapter, we will further formalize this diagrammatic scheme, and provide half-way of a *local proof of pragmaticism* in the language of gamma existential graphs. For the moment, it is interesting to notice that such a diagrammatic scheme is in complete accord with a category-theoretic perspective (in the sense of the mathematical theory of categories): the sign is relatively "free" (left of the diagram) until it incarnates in "concrete" environments (center of the diagram: interpretants $s_i,...,s_k,...$) and is later "functorially" reintegrated through pragmatic glueings (right of the diagram). The "one" (s) can truly enter a dialectical semiosis with the "many" (s_n). Peirce's pragmaticist maxim can be seen as a firm "bedrock" underlying an outstanding logico-semiotic abstract differential and integral world-view.

Phaneroscopy –or the study of the "phaneron", that is the complete collective present to the mind– includes the doctrine of Peirce's cenopythagorean categories, which study the universal modes (or "tints") occurring in phenomena. Peirce's three categories are vague, general and indeterminate, and can be found simultaneously in every phenomenon; they are further prescised and detached, following a recursive separation of interpretative levels, in progressively more and more determined contexts. Since they are *general* categories, their indetermination is mandatory (allowing them to incarnate "freely" in very diverse contexts), and their description is *necessarily* vague:

> The first is that whose being is simply in itself, not referring to anything nor lying behind anything. The second is that which is what it is by force of something to which it is second. The third is that which is what it is owing to things between which it mediates and which it brings into relation to each other.[112]

Peirce's *Firstness* detects the immediate, the spontaneous, whatever is independent of any conception or reference to something else:

> The first must be present and immediate, so as not to be second to a representation. It must be fresh and new, for if old it is second to its former state. It must be initiative, original, spontaneous, and free; otherwise it is second to a determining cause. It is also something vivid and conscious; so only it avoids being the object of some sensation. It precedes all synthesis and all differentiation; it has no unity and no parts. It cannot be articulately thought: assert it, and it has already lost its characteristic innocence; for assertion always implies a denial of something else.[113]

Secondness is the category of facts, mutual oppositions, existence, actuality, material fight, action and reaction in a given world. Secondness, with its emphasis on direct contrasts, balances the intangibility of firstness, closer to ungraspable intuitions (Joyce's epiphanies, Proust's Hudimesnil trees, Leibniz's monads). The conflict which characterizes experience is evident in the second category:

> The second category, the next simplest feature common to all that comes before the mind, is the element of struggle. This is present even in such a rudimentary fragment of experience as a simple feeling. For such a feeling always has a degree of vividness, high or low; and this vividness is a sense of commotion, an action and reaction, between our soul and the stimulus. (...) By struggle I must explain that I mean mutual action between two things regardless of any sort of third or medium, and in particular regardless of any action.[114]

Peirce's *Thirdness* proposes a mediation beyond clashes, a third place where the "one" and the "other" enter in dialogue. It is the category of sense, representation, synthesis, knowledge:

> By the Third, I understand the medium which has its being or peculiarity in connecting the more absolute first and second. The end is second, the means third. A fork in the road is third, it supposes three ways. (...) The first and second are hard, absolute, and discrete, like *yes* and *no*; the perfect third is plastic, relative, and continuous. Every process, and whatever is continuous, involves thirdness. (...) Action is second, but conduct third. Law as an active force is second, but order and legislation third. Sympathy, flesh and blood, that by which I feel my neighbor's feelings, contains thirdness. Every kind of sign, representative, or deputy, everything which for any purpose stands instead of something else, whatever is helpful, or mediates between a man and his wish, is a Third.[115]

Summing up, Peirce's vague categories can be tinctured with key-words as following:

> (1) *Firstness*: immediacy, first impression, freshness, sensation, unary predicate, monad, chance, possibility.
> (2) *Secondness*: action-reaction, effect, resistance, alterity, binary relation, dyad, fact, actuality.
> (3) *Thirdness*: mediation, order, law, continuity, knowledge, ternary relation, triad, generality, necessity.

The three peircean categories interweave recursively and produce a nested hierarchy of interpretative modulations (modes, tones or tints). The richness of Peirce's method lies in the permanent *iterative possibility* of his categorical analysis, a possibility which allows, in each new interpretative level, further and further refinements of previous distinctions obtained in prior levels. Knowledge –

understood as a progressive prescision (yielding thus progressive precision)– can grow defining more and more contexts of interpretation, and emphasizing in them some cenopythagorean tinctures.

The conceptual and practical back-and-forth between diverse layers is governed by the pragmatic maxim, which intertwines naturally with Peirce's categories. The maxim affirms that we can only attain knowledge after conceiving a wide range of representability possibilities for signs (firstness), after perusing active-reactive contrasts between sub-determinations of those signs (secondness), and after weaving recursive information between the observed semeiosis (thirdness). The maxim acts as a *sheaf* with a double support function[116] for the categories: a contrasting function (secondness) to obtain local distinctive hierarchies, a mediating function (thirdness) to unify globally the different perspectives. As we will emphasize later, an appropriate support for the good running of such a sheaf mechanism lies in a *continuity hypothesis*, according to which the permanent back-and-forth of signs and of their conceivable effects permeates all boundaries and crosses all cultural and natural environments.

Peirce's sign is a vague[117], general and undetermined triad, which gets bounded and sub-determined in progressive contexts. The most general form of a sign can be seen as a variant of a generic substitution principle: a sign is "something which *substitutes* something *for* something"[118]. Diagrammatically:

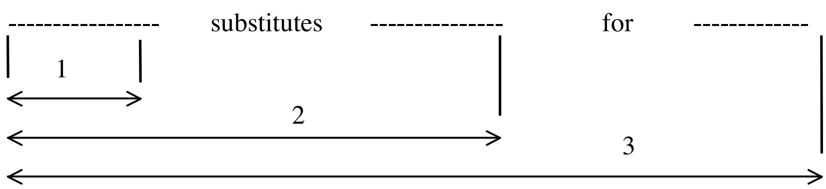


Figure 11.
Peirce's general sign

In a similar form, a "free" decomposition of "being" as a general sign can be represented in the following diagram, where Peirce's categories and the first levels of semiosis and modalization[119] become interweaved:

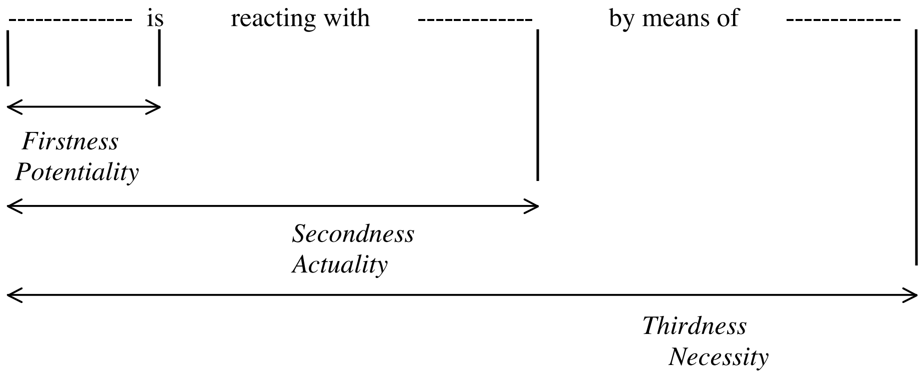


Figure 12.
"General sign" of Peirce's three categories

In Peirce's analysis, signs are *always* triadic. If, in some cases, a sign can be seen as dyadic, it is because triadicity has degenerated[120] in a combination of seconds. A first level of triadicity is found in the very definition of sign as a ternary generic relation S(-, -, -): –1– substitutes –2– for –3–. Term "2" is the "object" of the sign; term "1", which substitutes the object, is its "representamen"; term "3" is the medium, the interpretation context, the "quasi-mind" where the substitution is carried; *inside* that quasi-mind, the representamen acquires a new form: the "interpretant". A second level of triadicity –sub-qualifying the three ways in which object and representamen can correlate– produces Peirce's well-known initial classification of signs: icon (1), index (2) and symbol (3). An icon substitutes a given object: it signals a syntactic mark. An index is an icon which, furthermore, detects some changes of the object: it signals a semantic variation. A symbol is an index which, furthermore, weaves variations along an interpretation context: it signals a pragmatic integration. All sort of other sub-determinations are possible and the taxonomy can be refined recursively; Peirce came to distinguish at least 66 specific classes of signs.

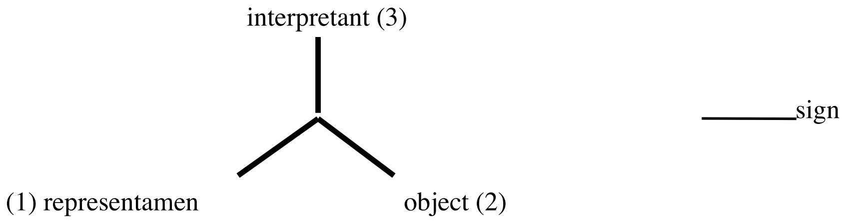


Figure 13.
Peirce's triadic sign

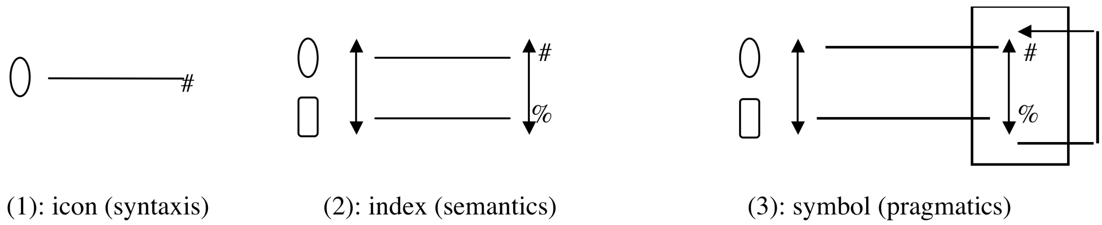


Figure 14. A Peirce triad.
Icon (sign in firstness), index (sign in secondness), symbol (sign in thirdness).

Logic, or universal semiotics, studies arbitrary transformations of signs and becomes a general theory of representations. Logic can then be seen as a sort of *geographical* science, which studies characters common to classes of "cognitive places", emphasizing semantic, *topographic* aspects (map designs including relative heights of each fixed cognitive place), as well as pragmatic, *projective* aspects (projection designs allowing comparisons of variable cognitive places). The construction of cognitive places profits from a multitude of representation processes, thanks to which mixed sensorial and formal data are recorded. With a complex machinery of logical *filters and lens*, the choice of interpretation contexts and the data insertion are controlled and its due relevance assured.

Adopting the pragmatic maxim, logic –understood as a projective and topographical science of cognitive places– includes an arsenal of tools to symbolize, contrast, follow and transfer information (some of these tools were reckoned in our previous chapter). Between different representations one can distinguish *implicit* relations (still not detected, potential) or *explicit* relations (already detected, actual).

An important objective in logic is to *turn explicit the implicit*, or, otherwise said, to actualize coherently the field of possible relations between representations. The interpretation practice is open-sided and extends to infinity, while new connections between representations are been captured. Connaturally with that unfolding and *continuous* semeiosis, logic has to deal with *general and global* tools, which cannot be reduced to purely existential or local considerations.

One of the strengths and major appeals of Peirce's semeiotics is to let *free* the notion of "quasi-mind", or interpretation context, where the semeiosis occurs (the "objects" are also very arbitrary: they can be physical objects, concepts, or any kind of signs where the semeiosis can again begin). Freeing interpretation environments from the psychologist shades related to a human "mind", Peirce's semeiotics turns unstoppably to a very wide range of universality. Since a quasi-mind can be either a protoplasm medium where semeiosis grows in back-and-forth processes of liquefaction and cohesion[121], or a nervous system where semeiosis integrates cells excitation, fibers transmission and habit taking, or a cultural environment spanned by linguistic grids, or even the very cosmos where the laws of physics are being progressively determined, it is clear that Peirce's "general signs" can cover huge domains of reality[122]. In that gigantic range, it is reasonable to abduct –as Peirce did– a possible *evolution* of signs towards determination:

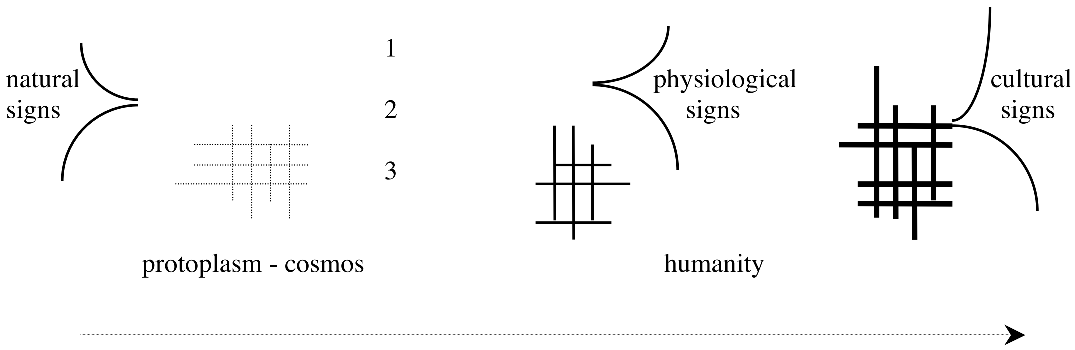
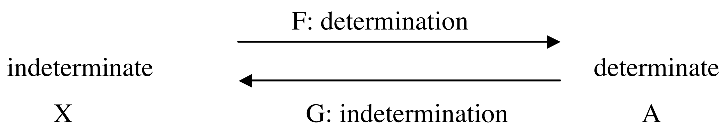


Figure 15.
"Progressive determination" of signs

Peirce's architectonics postulates a "dialectics" between indetermination and determination, opposing processes of progressive determination –a general evolutive tendency of signs in the universe– to the constant appearance of elements of indetermination and chance ("tychism") that periodically *free* the signs from their sedimentary semantic load. This back-and-forth between freeness and particularity, between generality and experience, between possibility and actuality, can be viewed in fact as a beginning of a *natural adjunction* between indetermination and determination:



Given an indeterminate sign X, its "concrete determination" FX is compared with another determinate sign A *in the same structural way* as its "free indetermination" GA can be compared with X:

$$[\ FX\ ,\ A\ ]\ \approx\ [\ X, GA\ ]\ .$$

Figure 16.
Adjunction between determination and indetermination

The "adjunction" bounds unitarily the dialectic back-and-forth: *determine partially the undetermined – undetermine partially the determined.* It is a double process of *saturation and freeness* which seems to govern not only many fundamental constructions in mathematics (from where the term "adjunction" is here borrowed), but also many basic information transfers in the cosmos. The *iterated* back-and-forth FG, FGF, FGFG,... produce in fact the great richness of Peirce's semeiotics: the accumulating spiral of undetermined and determined layers supports the "unlimited semeiosis" that refines without end our world conception. In any interpretability environment (that is, when interpretants and contexts of interpretation are *conceived*), many elements of "pure chance" undetermine what is apparently achieved, and other "saturation" tendencies determine what is apparently vague. In Peirce's words,

> In the beginning was nullity, or absolute indetermination, which, considered as the possibility of all determination, is being. A monad is a determination per se. Every determination gives a possibility of further determination. When we come to the dyad, we have the unit, which is, in itself, entirely without determination, and whose existence lies in the possibility of an identical opposite, or of being indeterminately over against itself alone, with a determinate opposition, or over-againstness, besides.[123]

> It is impossible that any sign whether mental or external should be perfectly determinate. If it were possible such sign must remain absolutely unconnected with any other.[124]

> We are brought, then, to this: conformity to law exists only within a limited range of events and even there is not perfect, for an element of pure spontaneity or lawless originality mingles, or at least must be supposed to mingle, with law everywhere. Moreover, conformity with law is a fact requiring to be explained; and since law in general cannot be explained by any law in particular, the explanation must consist in showing how law is developed out of pure chance, irregularity, and indeterminacy.[125]

Peirce's basic horizontal adjunction between generality and vagueness (studied in our first chapter), together with the transversal adjunction between determination and indetermination, shape together a planar grid where many peircean insights obtain an *orientation*. In most of Peirce's approaches to knowledge or nature, are *combined* –over a continuous bottom supporting osmotic passages– contrasting elements of indetermination, freeness and isolation with processes of determination, saturation and mediation. The many overlapping grids and layers which thus evolve in Peirce's architectonics guarantee the malleability of the edifice.

Peirce's categories permanently overlap in the phaneron. Phenomena are never isolated, never wholly situated in some detached categorical realm. Nevertheless, some readings can emphasize determined categorical layers, and can help to obtain important relative distinctions (the method shows, right away, that no absolute characterization is to be expected). Throughout his life, Peirce proposed more than one hundred of such layered readings in reference to the classification of sciences. In 1903, using his categories, Peirce came up with a lasting classification that Beverley Kent has designated as "perennial" classification[126].

The first recursive branching of the classification shows the places of mathematics and the *continuum*. Mathematics (1), ever-growing support of an ever-growing cathedral, emphasizes *possibilia*: it studies the abstract relational realm without any actual or real constraints. In place 1.1 of the classification, the mathematical study of the *immediately* accessible is drawn: the study of finite collections. In place 1.2, the study of mathematical *action-reactions* on the finite is undertaken: colliding with the finite, the infinite collections appear. In place 1.3 a *mediation* is realized: the general study of continuity appears. The awesome richness of mathematics arises from its peculiar position in the panorama of knowledge: constructing its relational web with pure possibilities, it reaches nevertheless actuality (and even reality) by means of unsuspected applications, guaranteeing in each context its necessity. The fluid wandering of mathematics –from the possible to the actual and necessary– is specific of the discipline.

Philosophy (2) is far from pure *possibilia* and closer to what is "given": it studies common phenomena to the general realms of experience (action-reaction over "existence" and potential "being"). Phaneroscopy (2.1) deals with universal phenomena in their firstness, in their immediacy, utilizing mathematical tools obtained in (1). Normative sciences (2.2) study common experiential phenomena, but from a secondness viewpoint: action of phenomena on communities, and action of communities on phenomena. Esthetics (2.2.1) studies impressions and sensations (firstness) produced by phenomena, consistently with an adequate "general ideal" (*summum bonum*); the "general ideal", that we will describe shortly, depends strongly on the *continuum*. Ethics (2.2.2) studies action-reaction (secondness) between the *summum bonum* and communities, giving rise to normative actions by communities in order to mate properly the "ideal". Logic (2.2.3) studies the mediating structures of reason (thirdness), coherently with the "general ideal". As Richard Robin has pointed out[127], the pragmatic maxim lies in a very interesting equilibrium point (2.2.3.3) in the classification, supporting the classificatory sciences which stand above the maxim and profiting from the particular observations of special sciences which lie under it. A more detailed study of this situation is undertaken in our next chapter, where we contend that a *continuous* interpretation of the "perennial" classification (in the language of gamma existential graphs) provides new clues to the central situation (2.2.3.3) of the pragmatic maxim.

- 1. Mathematics
  - 1.1. Finitude
  - 1.2. Infinitude
  - 1.3. *CONTINUUM*
- 2. Philosophy
  - 2.1. Phaneroscopy --- Three Categories
  - 2.2. Normative Sciences
    - 2.2.1. Esthetics
    - 2.2.2. Ethics
    - 2.2.3. Logic
      - 2.2.3.1. Grammar
      - 2.2.3.2. Critics
      - 2.2.3.3. Methodeutics
  - 2.3. Metaphysics
    - 2.3.1. Relative Ontology
    - 2.3.2. Physical Metaphysics --- Cosmology
    - 2.3.3. Religious Metaphysics --- Theology
- 3. Special Sciences
  - 3.1. Physics
  - 3.2. Psychics
    - Psychology – Sociology - Economics
    - Linguistics - Ethnology
    - History - Critics (...)
  - 3.3. "Systemics" (...)
- (...)

Figure 17.
Triadic "perennial" classification of sciences

Peirce showed that the "general ideal", according to pragmaticist requirements, could not be fixed, but *evolving*; that it could not be determined, but *open*; that it could not be particular, but *general*. Peirce's "general ideal" can then be described as the "continuous growing of potentiality". Accordingly, logic –which studies partial determinations of the "general ideal" in phenomenal thirdness– creates an evolving arsenal of relational and representational tools, searching specifically an accurate control on mediation and continuity processes. It is not thus surprising that

Peirce's advances in logic further evolved towards the construction of general "logics of continuity", such as the beta and gamma existential graphs systems.

One of the more significative forms of Peirce's triad is its modal decomposition: possibility as firstness, actuality as secondness, necessity as thirdness (see note 10). The systematic introduction of *possibilia* in any consideration can be seen as one of the great methodological strengths of Peirce's architectonics, and, in particular, of its pragmaticist maxim (after the "hard diamond" *mea culpa*). A full modalization of the maxim is, at bottom, what distinguishes the richness of Peirce's pragmaticism from other brands of pragmatism. Peirce's *continuum* –understood as a synthetical bondage place– is the pure field of possibility: as we have seen, the usual analytical decomposition ("points", "atoms") is supermultitudinously compacted, the units loose their actual singularity and particularities "blend" in a general realm. Modalization considerably enlarges Peirce's system and guarantees the appropriate multifunctionality of its architectonics.

III.2. The Continuum and Peirce's Architectonics

In many places of his work[128], Peirce insisted that the understanding of the *continuum* and the study of continuity formed one of the key problems in philosophy. For Peirce, continuity is an "indispensable element of reality"[129], that allows the development of evolutionary processes and that can be found in all realms of experience, from the liquid *continuum* which allows protoplasmic mutation, to the cosmic *continuum* which allows the expansive explosion of the universe, going through the *continuum* which underlies human thought and sensibility. Peirce baptized *synechism* a major thread in his philosophy that postulated a *real operativeness of continuity* in the natural world:

> The word *synechism* is the English form of the Greek συνεχισμοζ, from συνεχηζ, continuous. (...) I have proposed to make *synechism* mean the tendency to regard everything as continuous. The Greek word means continuity of parts brought about by surgery. (...) I carry the doctrine so far as to maintain that continuity governs the whole domain of experience in every element of it[130].

Synechism is closely weaved with the five structural arches (maxim, categories, logic, adjunction, classification) that support Peirce's architectonics[131]. A continuity principle is used in at least two crucial ways to insure the good running of Peirce's *pragmatic maxim*. First, one of the central ideas of pragmatism –namely, that every semiotic distinction *can be measured* in some way, through conceivable contrastable effects– finds its continuum expression in the statement that synechism guarantees the measurability of difference:

> Synechism denies that there are any immeasurable differences between phenomena.[132]

In fact, the pragmatic maxim postulates that two general signs (objects or concepts) are identical if and only if all their action-reactions in all conceivable interpretation contexts coincide, or, equivalently, that they are different if and only if some distinction can conceivably be measured between their diverse effects in the phaneron. Since in Peirce's *continuum* all differences can possibly be measured (using the *possibilia* monad around each "point"), the assumption of a general *continuum*, really operative in nature and close to Peirce's *continuum*, provides a strong backing to the maxim.

Second, only a continuous bottom can guarantee the semiotic overlappings, the gradual differential changes of tinctures and modalities, and the subsequent crucial integration processes that the pragmatic maxim requires for its exact functioning. *Only a continuum can anchor differences and analytic breakings, and –simultaneously– construct integrals and synthetic visions*. The peculiar strength of the pragmatic maxim –its *simultaneous* differential and integral character– lies thus on the *continuum*. Even deeper, only a *continuum* like Peirce's generic[133] and modal *continuum* –"all whatever is possible"[134]– can distinguish and reintegrate again all *possibilia realms* on which is based the full modalization of the maxim.

The three *cenopythagorean categories*, in one of Peirce's finest statements, may be understood as conceptual "tints", as gradual "tones" in the phenomenal *continuum*:

> Perhaps it is not right to call these categories conceptions; they are so intangible that they are rather tones or tints upon conceptions[135].

The tones or tints ("tinctures" in a existential graphs partial modelling) are modes, degrees, partial veils, that unfold over the *continuum* (musical, visual, schematic). Even if each fixation or analysis of those modes, each slip of the veils, means a discontinuity forced on space in order to partially represent it, the *totality* of those modes is fused in an unbreakable connection underlying the phaneron. The *prescision* used by Peirce to detach partially the categories[136] is no more than a methodological tool to partially decompose the *continuum*, a decomposition only offered to construct again new synthesis:

> Without continuity parts of the feeling could not be synthetized; and therefore there would be no recognizable parts[137].

Explicitly, in at least one sentence, Peirce states that the philosophy of continuity *leads* to triadic thought:

> The philosophy of continuity leads to an objective logic, similar to that of Hegel, and to triadic categories. But the movement seems not to accord with Hegel's dialectic, and consequently the form of the scheme of categories is essentially different[138].

In fact, Peirce's "movement" is not just linear: it can be viewed as a much more intertwined motion, closer to the *recursiveness* of Peirce's architectonics. A relative *back-and-forth* spiral process between continuity and triadicity takes place, and diverse evolutive contrasts detach in a *correlative* way (never a foundational or absolute one) the meaning of terms and the co-relations of concepts. Peirce's One, Two and Three serve as *ubiquitous* categories for tincturing all thought and nature, as formal bridges that overlap all *continuous* universe and humanity. Indeed, the human being is seen by Peirce as an iterated reflection of the categories, either in the physiological basis of its nerve cells (1: "disengaging energy"; 2: "nerve-currents"; 3: "acquiring habits")[139], or in the categories of his conscience (1: "feeling"; 2: "resistance"; 3: "synthetic consciousness")[140], or in the faculties of his psyche (1:

"pleasure"; 2: "desire"; 3: "cognition")[141]. The *continuum* of Peirce's categories, extended all over the phaneron, inscribes[142] itself in the line of medieval correspondences between micro and macrocosmos –in turn, evolved images of Pythagorean thought[143]– and can be seen as a modern form of the "Great Chain of Being", a universal scale of all existence, governed by a completeness principle (all possibility can be actually realized), a gradation principle (all actuality can be necessarily relativized), and a continuity principle (all necessity can be possibly glued).[144]

Logic (or *universal semeiotics*) is Peirce's par excellence tool to study systematically the multiple tones of the *continuum*. Peirce's logic, closer in its beginnings to boolean algebra, grows rapidly beyond its initial dualistic approach, and sets the way to a full logic of continuity, narrowly tightened with relative logic:

> The dual divisions of logic result from a false way of looking at things absolutely. Thus, besides affirmative and negative, there are really probable enunciations, which are intermediate. So besides universal and particular there are all sorts of propositions of numerical quantity. (...) We pass from dual quantity, or a system of quantity such as that of Boolian algebra, where there are only two values, to plural quantity.[145]

> While reasoning and the science of reasoning strenuously proclaim the subordination of reasoning to sentiment, the very supreme commandment of sentiment is that man should generalize, or what the logic of relatives shows to be the same thing, should become welded into the universal continuum, which is what true reasoning consists in.[146]

> Continuity is simply what generality becomes in the logic of relatives.[147]

> The continuum is that which the logic of relatives shows the true universal to be.[148]

Peirce signaled often that generality and continuity stood very close, as full forms of thirdness. The last two citations predicted that, on one side, generality could be interweaved to continuity, and, on the other side, that the webbing filter between them could be seen as the logic of relatives. As we showed in our previous chapter, these most intriguing and profound insights become in fact fully illuminated and corroborated by new findings in contemporary mathematical logic, proving again that the presence of a *continuum* underlying Peirce's architectonics is a key vault of the edifice. Also, far from being a "curiosity", Peirce's existential graphs – badly understood by peircean scholarship and grossly ignored by historians of logic, but, nevertheless, one of the most extraordinary blends of logic and continuity yet constructed– become then a vital arch of the architectonics. As Peirce was well aware calling his graphs "my chef d'oeuvre" (and as we will show in detail in our next chapter) most of the characteristic features of Peirce's architectonics –and, in particular, the essential place of the *continuum*– can be *fully reflected* in the behaviour of Peirce's systems of existential graphs.

In any case, it is also patent that, in order to obtain an adequate understanding of the *continuum*, several *reflections* of continuity should be handled, in *recursive and evolving* layers of growing complexity (corresponding, in part, to the more technical reflexivity properties of Peirce's *continuum*). In Peirce's words:

> Looking upon the course of logic as a whole we see that it proceeds from the question to the answer -- from the vague to the definite. And so likewise all the evolution we know of proceeds from the vague to the definite. The indeterminate future becomes the irrevocable past. In Spencer's phrase the undifferentiated differentiates itself. The homogeneous puts on heterogeneity. However it may be in special cases, then, we must suppose that as a rule the continuum has been derived from a more general continuum, a continuum of higher generality.[149]

Peirce's *indetermination-determination adjunction* is yet another example showing how some continuity considerations must be set in a hierarchy of levels and meta-levels. Over the meta-level of a meta-generic *continuum* ("continuum of higher generality") can schematically be drawn a *lower* (i.e. locally multi-layered) back-and-forth between *tychism* and *synechism* which pervades Peirce's architectonics:

> Permit me further to say that I object to having my metaphysical system as a whole called Tychism. For although tychism does enter into it, it only enters as subsidiary to that which is really, as I regard it, the characteristic of my doctrine, namely, that I chiefly insist upon continuity, or Thirdness, and, in order to secure to thirdness its really commanding function, I find it indispensable fully [to] recognize that it is a third, and that Firstness, or chance, and Secondness, or Brute reaction, are other elements, without the independence of which Thirdness would not have anything upon which to operate. Accordingly, I like to call my theory Synechism, because it rests on the study of continuity. I would not object to Tritism.[150]

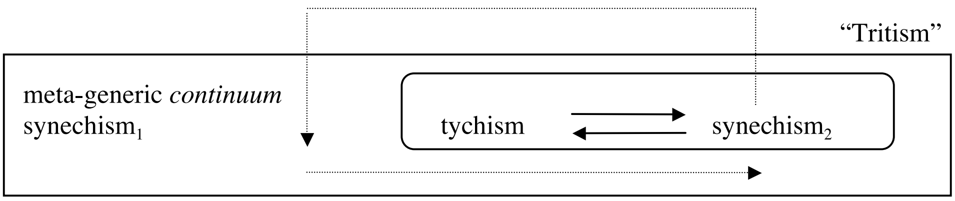


Figure 18.
Tychism-synechism adjunction drawn over a generic continuum

The introduction of elements of "pure chance" –the characteristic indetermination of tychism– is seen thus as a contextual *ingredient* inside a much more general process, where the primacy of the *continuum* is not contested. Indeed, the *continuum* happens to be the only truly generic concept on which the "design" of Peirce's architectonics can be sketched, since it is the only one which allows multiple intra-level internal reflections in the edifice. This explains Peirce's (otherwise cryptic) motto:

> Tychism is only a part and corollary of the general principle of Synechism.[151]

Peirce's triadic *classification of the sciences* extends also over a general *continuum*, which allows appropriate trifurcations of the *neighbourhoods* of the classification[152], encouraging translations, iterations and deiterations from one environment of knowledge to the other. The *continuum* not only supports the (possibility, actuality and necessity) of the transfers: deeper, it *induces* them, folding and unfolding systematically the unity and multiplicity of knowledge, considering polyvalent culture and philosophy as *natural gradation problems over the continuum*:

> The whole method of classification must be considered later; but, at present, I only desire to point out that it is by taking advantage of the idea of continuity, or the passage from one form to another by insensible degrees, that the naturalist builds his conceptions. Now, the naturalists are the great builders of conceptions; there is no other branch of science where so much of this work is done as in theirs; and we must, in great measure, take them for our teachers in this important part of logic. And it will

be found everywhere that the idea of continuity is a powerful aid to the formation of true and fruitful conceptions. By means of it, the greatest differences are broken down and resolved into differences of degree, and the incessant application of it is of the greatest value in broadening our conceptions.[153]

In his classifications of the sciences, Peirce studies the "generation of ideas by ideas"[154], and insists that all classifications evolve and "must certainly differ from time to time"[155]. On the evolving *continuum* of culture are molded very diverse classifications, but always with a central objective: render gradations more precise and define discipline frontiers, to further allow their crossing and merging. Drawing together objectives and methods of research, it becomes then natural that the study of frontiers (and of *free* "general similarities"[156] standing beyond specifics) has to be achieved over the very *modal genericity* of the *continuum*, where particulars dissolve and differences "melt" in a superior contiguity. Inextricably tied with the arches of Peirce's architectonics, the *continuum* fuses with the structural tensors that support the edifice.

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[109] "How to Make Our Ideas Clear" [1878; CP 5.402].
[110] "Harvard Lectures on Pragmatism" [1903; CP 5.18].
[111] "Issues of Pragmaticism" [1905; CP 5.438].
[112] "A Guess at the Riddle" [1887-88; CP 1.356].
[113] *Ibid.* [1887-88; CP 1.357].
[114] "Lectures on Pragmatism" [1903; CP 1.322].
[115] "One, Two, Three: an evolutionist speculation" [1886; W 5,300-301].
[116] We intend here a "sheaf" in its mathematical sense (as we used it in the previous chapter). A sheaf is based in a double function, *both* analytical and synthetical, which may well explain its conceptual richness: the sheaf "differentiates" its basis space (points look like fibers) but, in turn, it "integrates" the fibers' unfolded space. The mathematical conditions of "diversifying" (presheaf) and "glueing" (sheaf) are precisely the conditions which allow a *conjugation* of analysis and synthesis.
[117] For a particularly bright analysis of the interrelations between semeiotic, vagueness and continuity see Rossella Fabbrichesi Leo, *Sulle tracce del segno*, Firenze: La Nuova Italia, 1986, and *Continuità e vaghezza*, Milano: CUEM, 2001.
[118] The medieval formula for a sign (*aliquid stat pro aliquo*: "something which substitutes something") is a "degenerate second" variant of Peirce's fuller triadic formulation. Peirce's turn introduces permanently a "third" *for* ("something which substitutes something *for* something"), paving the way to pragmatic semiotics.
[119] In secondness –category of action-reaction and facts– falls at once the range of actuality. In firstness –category of immediacy– falls the range of possibility, understood as that which has *not* yet been contrasted (secondness) or mediated (thirdness). In thirdness –category of mediation and order– falls the range of necessity, understood as modal ordering or normative mediation.
[120] Peirce distinguished "genuine" thirds (ternary relations irreducible to combinations of monadic and binary predicates) and "degenerate" thirds (ternary relations constructible from monads and dyads). For example, *1 is between 0 and 2* is a degenerate third (can be reduced to the conjunction: "1 is

bigger than 0" and "2 is bigger than 1"), but *1+2=3* is a genuine third (sum is a ternary irreducible relation).

121 "A Guess at the Riddle" - "Trichotomic" [1887-88; EP 1,284].

122 According to Peirce's system, signs can even cover *all* reality if we allow an understanding of pure chance occurrences as "degenerate" signs in the second degree.

123 "The Logic of Mathematics" [1896; CP 1.447].

124 "An Improvement on the Gamma Graphs" [1906; CP 4. 583].

125 "A Guess at the Riddle" [c.1890; CP 1.407].

126 Beverley Kent, *Charles S. Peirce. Logic and the Classification of Sciences*, Montreal: McGill - Queen's University Press, 1987. The entry 3.3 ("systemics") does not appear in Peirce. Nevertheless *systemics* –in Niklas Luhmann's sense: a lattice of recursive feedbacks between environments (potential places for hierarchical information) and systems (actual information hierarchies)– seems to complete the classification in a natural way.

127 Richard S. Robin, "Classical Pragmatism and Pragmatism's Proof", pp.145-146, in: Jacqueline Brunning, Paul Forster (eds.), *The Rule of Reason. The Philosophy of Charles Sanders Peirce*, Toronto: University of Toronto Press, 1997.

128 Some examples: "It will be found everywhere that the idea of continuity is a powerful aid to the formation of true and fruitful conceptions" [1878; W 3,278]. "Continuity, it is not too much to say, is the leading conception of science" [c.1896; CP 1.62]. "The principle of continuity, the supreme guide in framing philosophical hypotheses" [c.1901; CP 6.101].

129 "What pragmatism is" [1905; EP 2,345].

130 "Immortality in the light of synechism" [1893; EP 2,1].

131 For the best presentation yet available of Peirce's architectonics from the viewpoint of general continuity principles, see Kelly Parker, *The Continuity of Peirce's Thought*, Nashville: Vanderbilt University Press, 1998. Parker shows masterfully how Peirce´s system can be understood as a structural glueing of the skeletons (1) of his classifications of the sciences, the lattices (2) of his systems of logic and semeiotics, and the "mediating binding forces" (3) of his generic continuity principles. The "continuous quasi-flow" or "relational stream" of Peirce's thought emerges with enormous coherence. Nevertheless, in the presentation of Peirce's *continuum*, Parker still relies too much on an introduction of Peirce's ideas *as compared* to Cantor's, loosing somewhat the force of Peirce's independent, truly original, approach to the labyrinth of the *continuum*.

132 *Ibid*. [1893; EP 2,3].

133 Demetra Sfendoni-Mentzou, "Peirce on Continuity and the Laws of Nature", *Transactions of the Charles S. Peirce Society* XXXIII (1997), 646-678, recalls that in the scholastic idea of generality ("Generale est quod natum aptum est dici de multis") generality is intrinsically welded with multiplicity. Thus, continuity, understood by Peirce as inexhaustible possibility and multiplicity, becomes the quintessence of generality.

134 "Detached ideas continued and the dispute between nominalists and realists" [1898; NEM 4,343].

135 "One, Two, Three" [c.1880; CP 1.353].

136 *Ibid*.

137 "Minute Logic" [c.1902; CP 2.85].

138 "A Philosophical Encyclopaedia" [c.1893; CP 8 G-c.1893, p.285].

139 "One, Two, Three: Fundamental Categories of Thought and Nature" [1885; W 5,247].

140 *Ibid*. [1885; W 5,246].

141 *Ibid*.

142 Peirce, thorough reader, knew well his place: "They [First, Second, Third] are not my discovery; in special and unphilosophical forms, they are familiar enough. They are well-known in philosophy; and have formed the basis of more than one famous system, already. But I have my way of apprehending them, which it is essential to bring to the reader's mind" (in: "First, Second, Third" [1886; W 5,302-303]). Peirce's original way consisted in detaching and utmost simplifying the terms, thanks to his outstanding logical acuity, to further use them in all conceivable realms, thanks to his outstanding philosophical weaving.

[143] Peirce's categories are *cenopythagorean*: not pythagorean, nor neopythagorean, but "full of freshness, χαινο-pythagorean". Ms 899 (c. 1904). In: C.S. Peirce, *Categorie* (ed. Rossella Fabbrichesi Leo), Bari: Laterza, 1992, p.129.
[144] Arthur O. Lovejoy, *The Great Chain of Being. History of an Idea*, Cambdrige: Harvard University Press, 1936. The completeness, gradation and continuity principles appear in Lovejoy's introduction, but the statements here presented are based on a peircean (modal-symmetric-triadic) reading.
[145] "A Guess at the Riddle" [c.1890; CP 1.354].
[146] "On Detached Ideas in General and on Vitally Important Topics" [1898; CP 1.673].
[147] "What pragmatism is" [1905; CP 5.436].
[148] "Detached ideas continued and the dispute between nominalists and realists" [1898; NEM 4,343].
[149] "The Logic of Events" [1898; CP 6.191].
[150] *Ibid.* [1898; CP 6.202].
[151] "Letter to William James" [1897; CP 8.252].
[152] Such a "neighbourhood" reading of the classification is explained in our next chapter, and depends essentially on a continuous "deiteration" of the classification in the sense of Peirce's gamma graphs.
[153] "The Doctrine of Chances" [1878; CP 2.646].
[154] "A Detailed Classification of the Sciences" - "Minute Logic" [1902; CP 1.216].
[155] *Ibid.* [1902; CP 1.203].
[156] *Ibid.* [1902; CP 1.215].

Chapter IV

Existential Graphs and Proofs of Pragmaticism

In this final chapter, we will show how Peirce's system folds on itself and finds local reflections –provable, or, at least, well grounded– which correspond to the major global hypotheses of the system. In particular, we will study how the pragmaticist maxim (i.e., the pragmatic maxim fully modalized, support of Peirce's architectonics) can be technically represented in Peirce's existential graphs, a truly original logical apparatus, unique in the history of logic, well suited to reveal an underlying continuity in logical operations and to provide suggestive philosophical analogies. Further, using the existential graphs, we will formalize –and prove one direction of– a "local proof of pragmaticism", trying thus to explain the prominent place that existential graphs can play in the architectonics of pragmaticism, as Peirce persistently advocated. Finally, we will present a web of "continuous iterations" of some key peircean concepts (maxim, classification, abduction) which supports a "lattice of partial proofs" of pragmaticism.

IV.1. Existential graphs reflections inside Peirce's architectonics

Back-and-forth osmotic processes are fruitful companions in Peirce's architectonics. In fact, constructing local reflections of global trends can be seen as a consequence of the permanent crossing of structural arches in Peirce's system (pragmaticist maxim, categories, universal semeiotics, indetermination-determination adjunction, triadic classification of sciences), a weaving that produces natural communicating hierarchies and levels in the edifice[157]. In the next diagram we synthetize a fold of Peirce's global architectonics on some of its local fragments:

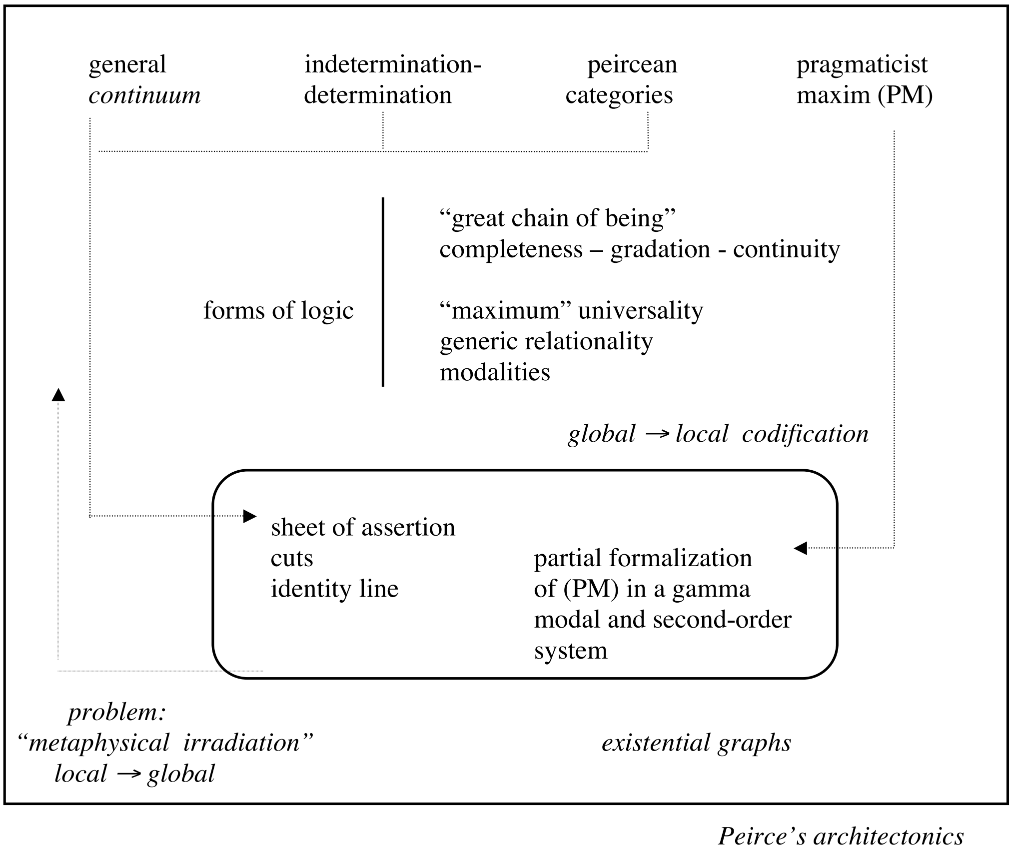


Figure 19.
Various level reflections of pragmaticist architectonics.
The global continuum inside the local continuum of existential graphs.
The modal form of the pragmatic maxim inside a system of gamma existential graphs

Peirce's systems of existential graphs –his "*chef d'oeuvre*" (Letter to Jourdain, 1908)– reflect *iconically* his entire philosophical edifice. The alpha sheet of assertion, continuous sheet on which the graphs are marked, stands as an iconic reflection of *real* non-degenerate continuity (thirdness), while the beta line of identity, continuous line which opens the possibility of quantifying portions of reality, stands as an iconic reflection of *existence* degenerate continuity (secondness):

> Since facts blend into one another, it can only be in a continuum that we can conceive this to be done. This continuum must clearly have more dimensions than a surface or even than a solid; and we will suppose it to be plastic, so that it can be deformed in all sorts of ways without the continuity and connection of parts being ever ruptured. Of this continuum the blank sheet of assertion may be imagined to be a photograph. When we find out that a proposition is true, we can place it wherever we please on the sheet, because we can imagine the original continuum, which is plastic, to be so deformed as to bring any number of propositions to any places on the sheet we may choose.[158]

> The line of identity which may be substituted for the selectives very explicitly represents Identity to belong to the genus Continuity and to the species Linear Continuity. But of what variety of Linear Continuity is the heavy line more especially the Icon in the System of Existential Graphs? In order to ascertain this, let us contrast the Iconicity of the line with that of the surface of the Phemic Sheet. The continuity of this surface being two-dimensional, and so polyadic, should represent an external continuity, and especially, a continuity of experiential appearance. Moreover, the Phemic Sheet iconizes the Universe of Discourse, since it more immediately represents a field of Thought, or Mental Experience, which is itself directed to the Universe of Discourse, and considered as a sign, denotes that Universe. Moreover, it [is because it must be understood] as being directed to that Universe, that it is iconized by the Phemic Sheet. So, on the principle that logicians call "the *Nota notae*" that the sign of anything, X, is itself a sign of the very same X, the Phemic Sheet, in representing the field of attention, represents the general object of that attention, the Universe of Discourse. This being the case, the continuity of the Phemic Sheet in those places, where, nothing being scribed, no *particular* attention is paid, is the most appropriate Icon possible of the continuity of the Universe of Discourse -- where it only receives *general* attention as that Universe -- that is to say of the continuity in experiential appearance of the Universe, relatively to any objects represented as belonging to it.[159]

> Among Existential Graphs there are two that are remarkable for being truly *continuous* both in their Matter and in their corresponding Signification. There would be nothing remarkable in their being continuous in either, or in both respects; but that the continuity of the Matter should correspond to that of Significance is sufficiently remarkable to limit these Graphs to two; the Graph of Identity represented by the Line of Identity, and the Graph of Coexistence, represented by the Blank.[160]

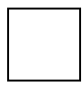
These quotes show the importance Peirce assigned to self-reference processes inside his system. Adequate symbolic concretions of the self-reference principle "*nota notae*" are observed both in the empty sheet of assertion and in the line of identity, graphs which continuously match their forms and meanings. Looking closely to the line of identity, Peirce analyzes further its full richness as a general sign, where iconical, indexical and symbolical tints blend together:

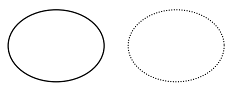
> The value of an icon consists in its exhibiting the features of a state of things regarded as if it were purely imaginary. The value of an index is that it assures us of positive fact. The value of a symbol is that it serves to make thought and conduct rational and enables us to predict the future. It is frequently desirable that a representamen should exercise one of those three functions to the exclusion of the other two, or two of them to the exclusion of the third; but the most perfect of signs are those in which the iconic, indicative, and symbolic characters are blended as equally as possible. Of this sort of signs the line of identity is an interesting example. As a conventional sign, it is a symbol; and the symbolic character, when present in a sign, is of its nature predominant over the others. The line of identity is not, however, arbitrarily conventional nor purely conventional. Consider any portion of it taken arbitrarily (with certain possible exceptions shortly to be considered) and it is an ordinary graph for which the figure "--is identical with--" might perfectly well be substituted. But when we consider the connexion of this portion with a next adjacent portion, although the two together make up the same graph, yet the identification of the something, to which the hook of the one refers, with the something, to which the hook of the other refers, is beyond the power of any graph to effect, since a graph, as a symbol, is of the nature of a *law*, and is therefore general, while here there must be an identification of individuals. This identification is effected not by the pure symbol, but by its *replica* which is a thing. The termination of one portion and the beginning of the next portion denote the same individual by virtue of a factual connexion, and that the closest possible; for both are points, and they are one and the same point. In this respect, therefore, the line of identity is of the nature of an index. To be sure, this does not affect the ordinary parts of a line of identity, but so soon as it is even *conceived*, [it is conceived] as composed of two portions, and it is only the factual junction of the replicas of these portions that makes them refer to the same individual. The line of identity is, moreover, in the highest degree iconic. For it appears as nothing but a continuum of dots, and the fact of the identity of a thing, seen under two aspects, consists merely in the continuity of being in passing from one apparition to another. Thus uniting, as the line of identity does, the natures of symbol, index, and icon, it is fitted for playing an extraordinary part in this system of representation.[161]

In fact, Peirce's line of identity can be considered fairly as the more powerful and "plastic" (in Peirce's *continuum* sense) of the symbolic conceptual tools that he introduced in the "topological" logic of existential graphs. Coherently with that plasticity, an adequate handling of a *thicker* identity line (existential quantifier in a second-order logic), will the basis of our approach[162] to a "local proof of pragmaticism". Next, we remind briefly[163] the basic properties of alpha, beta and gamma existential graphs needed to proceed.

Through a pragmatic collection of systems, the existential graphs cover classical propositional calculus (system of alpha graphs and generic illative transformations), first-order classical logic over a purely relational language (system of beta graphs and transformations related to the identity line), modal intermediate calculi (systems of gamma graphs and transformations related to the broken cut), and fragments of second-order logic, classes and metalanguage handlings (specific "inventions" of new gamma graphs). Over Peirce's *continuum* (generic space of pure possibilities), information is constructed and transferred through general action-reaction dual processes: *insertion – extraction*, *iteration – deiteration*, *dialectics yes-no*. The realm of Peirce's *continuum* is represented by a blank sheet of assertion where, following precise control rules, some cuts are marked, through which information is introduced, transmitted and eliminated. The diverse marks progressively registered in the sheet of assertion allow logical information to *evolve* from indetermination to determination, thanks to a precise triadic machinery: (1) formal graphical languages, (2) illative transformations, (3) natural interpretations, all well intertwined in a pragmatic perspective.

1. Signs.

Sheet of assertion: blank generic sheet. Icon: 

Cuts: generic ovals detaching regions in the sheet of assertion. Icons: 
(alpha) (gamma)

Line of identity: generic line weaving relations in the sheet of assertion. Icon: ________
(beta)

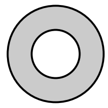
Logical terms : propositional and relational signs marking the sheet of assertion. Icons: p, q, ... R, S, ...

2. Illative Transformations of Signs.

Detaching Properties ("information zones").

Cuts can be nested but cannot intersect.
Identity lines can intersect other identity lines and all kinds of cuts.

Double cuts alpha can be introduced or eliminated around any graph, whenever in the "donut" region (gray) no graphs different from identity lines appear.



Transferring Properties ("information transmission").

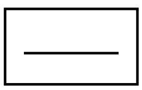
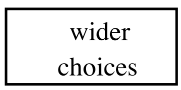
Inside regions nested in an even number of alpha cuts, graphs may be *erased*.
Inside regions nested in an odd number of alpha cuts, graphs may be *inserted*.
Towards regions nested in a bigger number of alpha cuts, graphs may be *iterated*.
Towards regions nested in a lower number of alpha cuts, graphs may be *deiterated*.

3. Interpretation of Signs and Illative Transformations.

Blank sheet:	truth
Alpha cut:	negation
Juxtaposition:	conjunction
Line of identity:	existential quantifier
Gamma cut:	contingency (possibility of negation)
Double cut:	classical rule of negation $(\neg\neg p \leftrightarrow p)$
Erasure and insertion:	minimal rule of conjunction $(p \wedge q \rightarrow p$ and $\neg p \rightarrow \neg(p \wedge q))$
Iteration and deiteration:	intuitionistic rule of negation as generic connective $(p \wedge \neg q \leftrightarrow p \wedge \neg(p \wedge q))$

Figure 20.
Rudiments of Existential Graphs

The existential graphs variety of formal languages and illative transformations can be turned into logical *calculi* if one assumes surprisingly elementary axioms:

• axioms:  (ALPHA)  (BETA) 

(GAMMA)

• *calculi*:

ALPHA	≡	Classical propositional calculus
BETA	≡	Purely relational first-order logic
GAMMA$_{I}$	≡	Intermediate modal logics[164]
GAMMA$_{II}$	⊇	Second-order logic.

Peirce hoped that the existential graphs could help to provide a full "apology for pragmaticism"[165]. In fact, in all due justice, the *very existential graphs looked at themselves* –under the perspective that Roberts' and Zeman's completeness proofs have supplied– provide an outstanding apology for the deep pragmatic approach that Peirce undertook in logic:

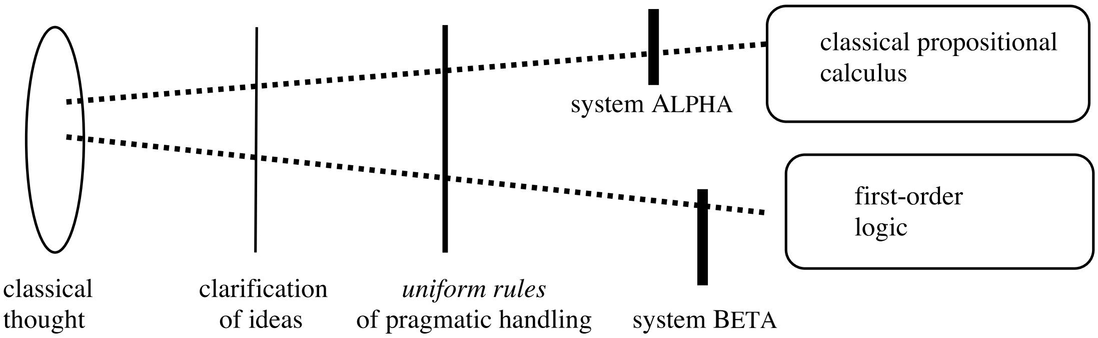
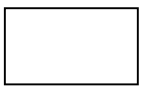
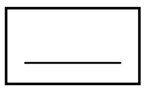
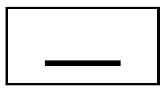


Figure 21.
Existential graphs as an "apology for pragmaticism"

Indeed, the *simultaneous* axiomatization of classical propositional calculus and purely relational first-order logic, with the *same five generic rules* (double alpha cuts, insertion, erasure, iteration and deiteration), renders explicit *technical common roots* for both *calculi* which have been ignored in all other available presentations of classical logic. The *same rules* detect, in the context of alpha language, the handling of classical negation and conjunction, and, in the context of beta language, the handling of the existential quantifier: something just unimaginable for any logic student raised into Hilbert-type logic systems. Thus –in agreement with Peirce's pragmatic maxim and Peirce's "idealist" realism– the ALPHA and BETA *calculi* show that there exists a *kernel*, a *"real general"* for classical thought, a kernel which, in some representational contexts, gives rise to the classical modes of *connection*, and which, in other contexts, gives rise to the classical modes of *quantification*. The common roots for classical connectives and quantifiers are revealed in *common* pragmatic action-reaction processes, global and general, which in *diverse* representational contexts generate derived rules, local and particular, proper to each context. We face thus a truly remarkable "revelation" in the history of logic, not yet fully understood nor valued in all its depth. It is, in a very precise way, the *only* known presentation of classical logical *calculi* which uses the same global and generic axiomatic rules to control the "traffic" of connectives and quantifiers.

In turn, the "apology for pragmaticism" obtained with the existential graphs shows the coherence of the synechist abduction, at least if it is restricted to the continuum underlying classical logic. In fact, the existential graphs show that the rules of classical connectives and quantifiers correspond continuously to each other over a generic bottom; their apparent differences are just contextual and can be seen as breaks on the underlying logical continuity. But even beyond the classical realm, as we hinted in our second chapter, we count on several mathematical supports to conjecture that the synechist hypothesis can span a wider range of validity, including –fair abduction– diverse progressive forms of the logical *continuum* (intuitionistic, categorical, peircean) up to –bold abduction– the cosmological *continuum*.

A pair of examples, where (going from local to global) we re-interpret some specific "marks" of the graphs, can be useful to show the possible interest of a "metaphysical irradiation" of the graphs. In first place, the immediate comparison of axioms for the ALPHA, BETA and GAMMA$_{II}$ (second order) *calculi*,

 (ALPHA)  (BETA)  (GAMMA$_{II}$)

shows symbolically that existence (first and second-order lines of identity) can be seen, *simultaneously*, as a continuity break in the "real general" (blank sheet of assertion), and as a continuity link in the "particular" realm (ends of the identity line). The identity lines, continuous sub-reflections of the sheet of assertion, are self-reflexively marked on the general *continuum* and allow to construct the transition “from essence to existence”[166]. The elementary axioms of the basic systems of existential graphs support thus the idea –central in philosophy (pre-socratics, Heidegger)– that a first self-reflection of “nothingness on nothing”[167] can be the initial spark that puts in motion the evolution of the cosmos.

In second place, the continuous iterations of lines of identity (beta or gama$_{II}$) through cuts (alpha or gamma) (see *figure 22*) show that existence is no more than a form to link continuously fragments of actuality inside the general realm of all possibilities. It would be fallacious, then, as Peirce severely advocated in his “disputes against nominalists”, to think the existent, the actual, the given, *without previously assuming* a coherent continuous bottom of real *possibilia*, a bottom needed in order to guarantee the *relational emergence* of existence:

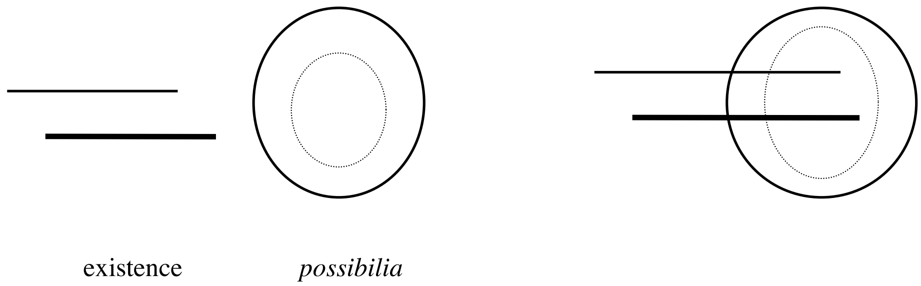


Figure 22.
Continuous iterations (and deiterations) of lines of identity.
Existence (actuality, secondness) is continuously linked to real possibilities.

IV.2. A local proof of pragmaticism

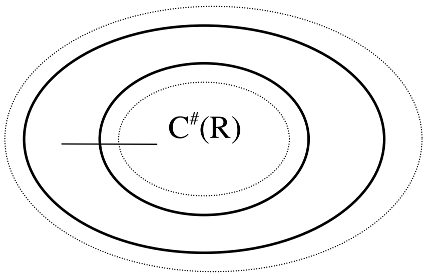
In 1903, in his Harvard conferences, Peirce thought he had guessed a “proof of pragmaticism”[168]. Of course, such a proof, in an absolute and global sense, could not be sustained and would go in opposite direction to the pragmatic maxim. Nevertheless, the impossibility of an absolute proof does not preclude that some fragmentary and local codings of the proof could, in principle, be realized. Peirce insisted that the existential graphs should help in that task, but it seems that he never fully completed the scattered indications left in his latter writings[169]:

> I beg leave, Reader, as an Introduction to my defence of pragmatism, to bring before you a very simple system of diagrammatization of propositions which I term the System of Existential Graphs. For, by means of this, I shall be able almost immediately to deduce some important truths of logic, little understood hitherto, and closely connected with the truth of pragmaticism.[170]
>
> You apprehend in what way the system of Existential Graphs is to furnish a test of the truth or falsity of Pragmaticism. Namely, a sufficient study of the Graphs should show what nature is truly common to all significations of concepts; whereupon a comparison will show whether that nature be or be not the very ilk that Pragmaticism (by the definition of it) avers that it is.[171]
>
> It is one of the chief advantages of Existential Graphs, as a guide to Pragmaticism, that it holds up thought to our contemplation with the wrong side out, as it were.[172]

We now present a translation of the "full modal form" of the pragmaticist maxim (*figure 10*, previous chapter) to the language of existential gamma graphs, indicating advances and limitations in our approach[173]. In particular, a formalization of the maxim, *half-way provable in a modal second-order gamma system*, shows that the maxim can acquire new supports for its validity. Indeed, beyond the clear usefulness of the maxim as a global philosophical method (abductively stated, inductively checked), it is also of precious value to count on a reflection of the maxim as a valid local theorem (deductively inferred). Peirce's pragmaticist maxim, always considered by Peirce as an hypothesis, obtains thus a new confirmation by means of a logical apparatus. The three dimensions of reasoning (abduction-induction-deduction) become strongly welded. If –in the future– the structural transfer from local to global fostered in part by pragmaticism becomes better understood, the local gamma proofs of pragmaticism could then acquire an unsuspected relevance to support the general architectonics of the system.

In first instance, combining the notion of "integral" (relational glueing) and the formalism of gamma graphs, we can obtain an intermediate, *semi-formal*, statement of the pragmaticist maxim. The value of semi-formality (or "informal rigour") consists in allowing further refinements, depending on the way the "integral" is afterwards rendered symbolically in adequate gamma systems[174]. An intermediate expression of the pragmaticist maxim is the universal closure of the following statement, obtained directly as a diagrammatic translation of the full modal form of the maxim to a "mixed" language with existential graphs (semi-formal "mixtures" involving symbols ≡ and ∫ will soon be deleted):

$$C \equiv \int_{R,\#}$$



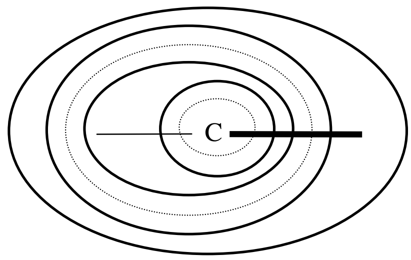
($PRAG_{EG}$),

that is: for all C, "C is equivalent to the integral of all necessary relations between interpretants of C and elements of their contexts, running on all possible interpretative contexts". With the usual logical symbols this can also be written semi-formally:

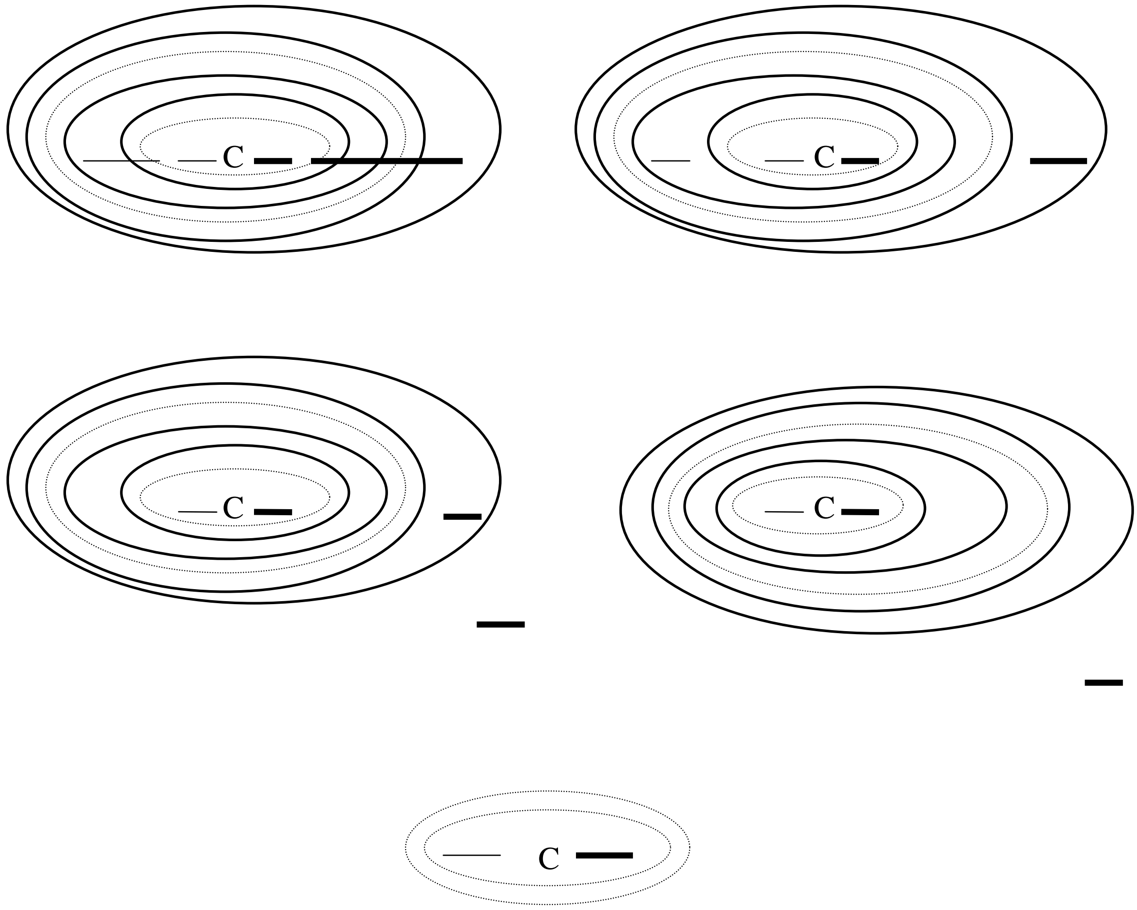
$$\forall C \ (\ C \equiv \int_{R,\#} \Diamond \exists x \ \Box \ C^{\#}(R,x) \ ) \ .$$

The pragmaticist maxim, understood semi-formally as the (universal closure of) the intermediate statement ($PRAG_{EG}$), can then be implemented locally in diverse gamma fully formal systems, in which ($PRAG_{EG}$) may become a *theorem* of the system. As the implementation will be more *faithful*, and the gamma system will be more *universal*, the pragmaticist maxim will acquire greater deductive strength. We proceed now to an *elementary* implementation of the maxim in a *specific* gamma system, closely related to Peirce's general realism (scholastic reality of universals, where the possibly necessary becomes actual). The implementation is still far from being duly faithful (codifies all interpretants in just one sign), and the gamma system is still away from true universality (requires the axiom $\Diamond\Box\, p \leftrightarrow p$), but we think that an important step in a local proof of pragmaticism is here undertaken.

Consider ($PRAG_{EG}$): $C \equiv \int_{R,\#} \Diamond \exists x \,\Box\, C^{\#}(R,x)$. Identifying # with *identity* (use of the self-reference principle "*nota notae*": codification of all interpretants of a sign in the sign itself), and translating the integral $\int$ as a *universal quantification* on all relations, we see that the right-hand side of ($PRAG_{EG}$) can be represented by the following diagram[175] (where the thicker line stands for a gamma second-order existential quantifier):



Now, using the rules of erasure, deiteration, and double alpha cut elimination, it is shown that this diagram (that we can call the "pragmatic reading of C") illatively *implies* the following diagrams[176]:



that is, the diagram representing the "pragmatic reading of C" does in fact imply C, *in the case in which the double broken cut may be erased*, that is when the modality $\Diamond\Box$ can be eliminated.

This shows that *one* of the two implications in the equivalence that constitutes a local form of the pragmaticist maxim (the "positive" implication according to which the pragmatic knowledge of C guarantees the knowledge of C) can be proved in systems in which $\Diamond\Box p \rightarrow p$, that is in systems in which the possibly necessary implies the actual. On the other hand, the reverse implication does not seem to be provable[177], not even in case we could count on *introducing* double broken cuts (corresponding to a full equivalence $\Diamond\Box p \leftrightarrow p$). We can call this reverse implication the "negative" one: the denial of one of the conceivable characters of C implies not-C. Arguably, this "negative" implication can be considered the more interesting one from the perspective of a fallibilist architectonics such as Peirce's, showing that our advance in the weaving *graphs-pragmaticism* is still a modest one. To obtain a fuller equivalence between C and its pragmatic reading, a finer implementation of the pragmaticist maxim would have to be achieved, but we hope our tentative opens the way in such a *possibilia* realm.

Our reflection of the global pragmaticist maxim –half-way provable in a local setting of gamma graphs– can be considered as a further indication (*induction*) of the eventual correction of the general maxim. Peirce had proposed the maxim as a hypothesis (*abduction*) to be criticized, contrasted, and refined. An important trend of research would then consist in obtaining other interesting implementations of the maxim that could become theorematic (*deduction*) in other gamma systems[178]. The vertical glueing of many theorematic implementations of the maxim would be very close to a wide "proof of pragmaticism".

IV.3. "Vague proofs" of pragmaticism

A sound use of the pragmaticist maxim –applied reflexively to itself in a self-unfolding *continuum*, helping to understand better its eventual "proof"– shows that arguments in favour of pragmaticism can never be set in a definitive way, in an absolute space. Indeed, as the maxim itself advocates, any argument that hopes to attain a certain degree of *necessity* has to be set *locally* in a determined interpretation context. From this elementary observation, it follows that the "proof of pragmaticism" sought by Peirce *may* (in fact, *must)* be seen as a sophisticated *lattice of partial proofs*, where along diverse hierarchical levels converge local abductions, inductions and deductions, which *may* (*must*) correlate each other, but that can never be summarized in a unique "transcendental deduction". Peirce's architectonics shows, in fact, that knowledge is always constructed along different perspectives, floors and levels –like Borges' Babel tower, doubly infinite, never comprised in a unique glance– without a "transcendental" or "absolute" vantage point from where a complete panorama could be stared at (observe that the non-existence of such a "point at infinity" is perfectly linked with the non-existence of privileged points in Peirce's *continuum*).

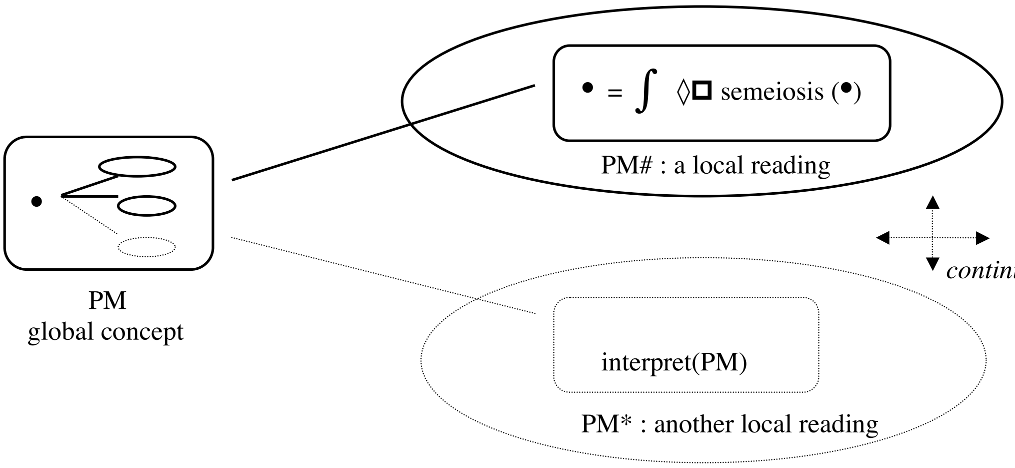


Figure 23.
The pragmaticist maxim (PM) applied to itself: PM(PM).
Infinite ramification of Peirce's architectonics.
Continuous lattice of local proofs of MP.

Inside Peirce's architectonics it is thus natural to emphasize some *argumentative mixtures* (confluences abduction – induction – deduction) which build up the lattice of supports for pragmaticism. It may be said that all of Peirce's work – from his first timid logical comments to his final daring cosmological speculations– consists in the meticulous and perseverant construction of that lattice, always trying to enlarge consistently its range of validity, to extend its depth and to correlate its diverse "marks". Of course, we face *a lattice of marks sketched over a continuous bottom*, where, once again, plays an extraordinary role the *natural correspondence* between a general philosophical trend, the world which supports it and the methods which seek to prove it. In the following, we will study just some of the marks supporting pragmaticism, which are closely related to the *continuum*: the existential graphs as "apology for pragmaticism", the central place of the pragmatic maxim in the classification of sciences, the self-referential and "fixed-point" arguments sustaining pragmaticism, and, finally, the "logic of abduction".

One of the finer marks in support of Peirce's pragmaticism is a natural "continuity interpretation"[179] of some peculiar features of the existential graphs. On one side, the genesis[180] of the graphs shows clearly that they were constructed *continuously*, departing from diagrammatic experiments related to the logic of relatives (letter to Mitchell, 1882; reply to Kempe, 1889), coming abductively to propose basic rules and ideas (entitative graphs, 1896), and making afterwards permanent corollarial illations, inductively contrasted and polished (entries in the *Logic Notebook*, from 1898 on), up to constructing truly theorematic *systems* of existential graphs (Alpha, Beta, Gamma, 1903). It is interesting to notice that this process of discovery uses fully the argumentative triad abduction – induction – deduction, and that it *only* uses that mixture. Since the result is the *simultaneous* re-construction of both classical propositional calculus and first-order logic, which can be considered as a neat basis for the main general qualitative and quantitative modes of thought, the construction of the existential graphs shows that Peirce's argumentative triad may include the *continuum of all possible types of arguments* representable in classical thought. In this way, the pragmaticist hypothesis stating that the triad abduction – induction – deduction *saturates* all inferential processes obtains an important backing: another "mark" in our lattice-type "proof of pragmaticism".

On the other side, the construction of the existential graphs should be understood as a full "apology" for pragmaticism and synechism, not only because of the unveiling of the "real general" for classical thought that we have already discussed, but also because of its ability to represent pragmatically –in its *language*, *rules* and *axioms*– deep local reflections of the global continuous trends present in the architectonics. The language of existential graphs reflects iconically the cosmological *continuum* (thirdness), its continuity breaks (secondness) and its chance elements (firstness): the alpha sheet of assertion and the beta line of identity are plastic fusion operators (thirdness), the alpha cuts are segmenting marks which depart from the real general and give rise to actual existence (secondness), the gamma cuts are fissures which open the way to chance and possibility (firstness). The rules, or illative transformations, reflect in an outstanding pragmatic way the more elementary osmosis occurring in semeiosis: registering and forgetting information (rules of insertion and erasure), detaching and transgressing dual information zones (rules of introduction and erasure of double alpha cuts), transferring and recovering information (rules of iteration and deiteration). Finally, the axioms, as already mentioned, can be thought as a nutshell expression of Peirce's wider general synechism.

If, following Peirce, we understand the pragmatic maxim as a part of "methodeutics" ("studying methods to be followed in the search, exposition and application of truth"[181]), its place in the "perennial" classification of sciences lies naturally in the trichotomic subdivision 2.2.3.3, a prominent central place inside the classification which supports generality layers above it and profits from particularization layers below, as Richard Robin has pointed out[182]. Going deeper, and *extending continuously* Robin's fundamental remark, we may understand pragmaticism as a continuous irradiation of the maxim –more precisely, as its *continuous iteration and deiteration– from place 2.2.3.3 towards all other neighbourhoods* of knowledge present in the classification:

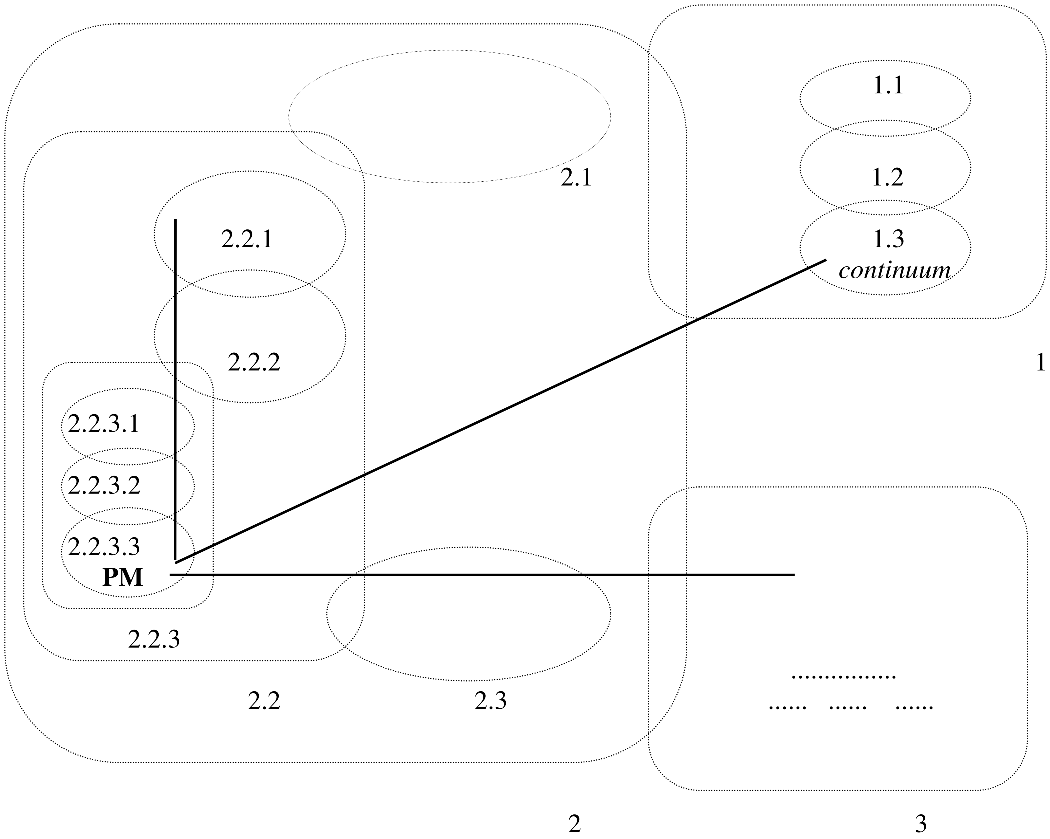


Figure 24.
Continuous iterations of the pragmatic maxim (PM)
along a continuous unfolding of the triadic classification of sciences

The previous diagram suggests another useful argument to consolidate the *global web of local marks* in which may consist the "proof of pragmaticism". The diagram suggests to construct an adequate *translation* of the classification into existential graphs, a translation which should perhaps be *inverse* (or done in a sheet *verso*) to the one represented in *figure 24* –where regions with *more* trichotomic ramifications in the classification tree should be surrounded by *less* cuts– in such a way that the pragmatic maxim could really be *iterated* towards all other neighbourhoods in the classification. An even finer implementation would have to introduce also the *types* of gamma cuts which should be nested *iconically* around fragments of the classification: possibility (broken-alpha) cuts for trichotomies of type 1, actuality (alpha) cuts for trichotomies of type 2, necessity (alpha-broken-alpha) cuts for trichotomies of type 3. If this kind of translation could be done, we could pass from *discrete* models for the classification (trees with ramification 3) to *continuous* models (assertion neighbourhoods, natural osmosis), producing thus a coherent sub-determination of Peirce's synechism. An effective continuous implementation of *figure 24* could also help to understand, not only the central irradiation of the maxim in all fields of knowledge, but also the natural pre-eminence of some crossings between disciplines *in detriment* of others, constructing thus the prolegomena of a true "topographical" science which could determine "heights" and "access roads" in the continuous relief of knowledge[183].

The central place of the pragmatic maxim in the classification of sciences allows to perceive the maxim as a *balance environment* in a wide structure. In turn, pragmaticism can also be understood as a generic *fixed-point* technique, a reflexive and self-referential apparatus which, through each self-application, stratifies the field of interpretation. Peirce's fourth article (1909) in the *Monist* series was going to present

> a theory of Logical Analysis, or Definition [which] rests directly on Existential Graphs, and will be acknowledged, I am confident, to be the most *useful* piece of work I have ever done... Now Logical Analysis is, of course, Definition; and this same method applied to Logical Analysis itself –the definition of definition– produces the rule of pragmaticism.[184]

Another fixed-point tentative to guarantee the unavoidable centrality of pragmaticism appears, as the editors of the *Essential Peirce* have well noticed, when Peirce, trying to characterize habit as a final logical interpretant, shows that habit can only be defined through other habits[185]:

> The deliberately formed, self-analyzing habit, –self-analyzing because formed by the aid of analysis of the exercises that nourished it–, is the living definition, the veritable and final logical interpretant. Consequently, the most perfect account of a concept that words can convey will consist in a description of the habit which that concept is calculated to produce. But how otherwise can a habit be described than by a description of the kind of action to which it gives rise, with the specification of the conditions and of the motive?[186]

In this way, habits turn out to be fixed-points of the self-referential operator *definition of the definition*, since its definition resorts to the very same term which is being defined. Now, the fact that habits can be seen as fixed-points connects again in a very natural way the architectonics of pragmaticism with its underlying *continuum*. Indeed, it can be shown in modern mathematics that, underneath *any* fixed-point theorem, lies a natural topology which renders *continuous* the fixed-point operator and which allows to construct the fixed-point as a *limit* of discrete approximations. The local results of modern mathematics, abductively and continuously transferred to the global design of the architectonics, provide thus another "mark" which pulls taut the web of supports of pragmaticism. For future endeavours remains the task of modelling –inside the mathematical theory of categories– an integral translation of some the differential "marks" we have been recording: the "free" iconicity of existential graphs, the iterative "universality" of the pragmatic maxim, the "reflexivity" of habits.

The pragmaticist maxim, fully modalized, depends crucially on a range of *possible* interpretation contexts, where some hypothetical representations are subject to further deductive inferences and inductive contrasts. Peirce's *logic of abduction* –understood as a system to orderly adopt hypotheses with respect to *given* contexts[187]– lies then at the very core of pragmaticism:

> If you carefully consider the question of pragmatism you will see that it is nothing else than the question of the logic of abduction. That is, pragmatism proposes a certain maxim which, if sound, must render needless any further rule as to the admissibility of hypotheses to rank as hypotheses, that is to say, as explanations of phenomena held as hopeful suggestions; and, furthermore, this is all that the maxim of pragmatism really pretends to do, at least so far as it is confined to logic (...) A maxim which looks only to possibly practical considerations will not need any supplement in order to exclude any hypotheses as inadmissible. What hypotheses it admits all philosophers would agree ought to be admitted. On the other hand, if it be true that nothing but such considerations has any logical effect or import whatever, it is plain that the maxim of pragmatism cannot cut off any kind of hypothesis which ought to be admitted. Thus, the maxim of pragmatism, if true, fully covers the entire logic of abduction.[188]

From the natural correlation "pragmaticism :: logic of abduction" it follows that another "vague" proof of pragmaticism –another mark in its supporting web– should be looked for in an adequate continuous understanding of the "logic of abduction", that is of the abductive inference, illation and decidability processes. In effect, as Peirce notices,

> It must be remembered that abduction, although it is very little hampered by logical rules, nevertheless is logical inference, asserting its conclusion only problematically or conjecturally it is true, but nevertheless having a perfectly definite logical form.[189]

Abduction's "perfectly definite logical form" arises in Peirce's early studies (1860's) around "vague" variations of the Aristotelean syllogism. We suggest (see *figure 25*) that already in those early researches the fundamental adjunctions "determinacy – indeterminacy" and "definition – vagueness" may have entered in Peirce's thought. In those beginnings, the adjunctions may have been only intuitive, plastic, *continuous* processes, but they may have allowed Peirce to bend the rigid Aristotelean rules and to jump to the *verso* of Peirce's logical creativity:

Variations on the syllogistic form *a i i* in the first figure

All X is Y Some Z is X ———— Some Z is Y	*deductive form*	*implicative inference* *general + vague ⇒ vague*
Some Z is Y Some Z is X ———— All X is Y	*inductive form*	*vague + vague ⇒ general* *(no inference)*
All X is Y Some Z is Y ———— Some Z is X	*abductive form*	*retro-implicative inference* *general + vague ⇒ vague*

Figure 25.
Syllogistic abduction as "vague" deformation of syllogistic deduction.

Understood as a *system* to provide reasonable hypotheses which could explain irregular states of things, Peirce's abduction develops between 1870 and 1910, accurately defining the system's tools in accordance with the general dictate of logic to evolve towards progressive determination. The "logic of abduction" refines Peirce's prior ideas on the "logic of discovery": its ability to undergo experimental testing, its capacity to explain surprising facts, its economy, its simplicity, its plausibility, its correlation with the evolved instinct of the species[190]. Led by his breakthroughs in the logic of relatives, Peirce moves from describing analitically the *particular predicative* form of syllogistic abduction towards constructing synthetically abduction as a *general relational* system: contextual and contrasting handling of hypotheses, optimization and decision "filters" to maximize the likelihood of adequate hypotheses, search of correlations between the complexity of hypotheses and their probability of correction.

Deductive systems

$$\frac{\Gamma \vdash \alpha \rightarrow \gamma}{\Gamma, \alpha \vdash \gamma}$$

Abductive systems

$$\frac{\Gamma \vdash \alpha \rightarrow \gamma}{\Gamma, \gamma \vdash \Diamond\alpha}$$

$$\Gamma, \gamma \vdash \text{Prob}(\alpha)$$

In general, there are important correlations between the conclusion's complexity *in context's eyes* (Γ-Compl(γ)) and the probability of the explanatory correction of the hypothesis (Prob(α)). The higher the complexity (Γ-Compl(γ)), the more plausible becomes the equivalence Prob(α) $\equiv \alpha$ along the context Γ, *reversing* thus the inference[191].

Figure 26.
Abduction as a system of logical approximation
towards correctness and optimization of explanatory hypotheses.

The logic of abduction, as Peirce himself mentions very precisely, tries to explain in a systematic way *regularity breaks* and homogeneity disorders, along given contexts, that go beyond simple casual (punctual) irregularities. In fact, explanation is only really needed when it goes *beyond particulars* and when it fuses into the general (the *continuum*):

> The only case in which this method of investigation, namely, by the study of how an explanation can further the purpose of science, leads to the conclusion

that an explanation is positively called for, is the case in which a phenomenon presents itself which, without some special explanation, there would be reason to expect would *not* present itself; and the logical demand for an explanation is the greater, the stronger the reason for expecting it not to occur was. (...) But if we anticipate a regularity, and find simple irregularity [irregularity being the prevailing character of experience generally], but no *breach* of regularity, –as for example if we were to expect that an attentive observation of a forest would show something like a pattern, then there is nothing to explain except the singular fact that we should have anticipated something that has not been realized.[192]

Abduction reintegrates breach and context from a higher perspective, and fuses them in a common explanatory *continuum*. Thus, the deep task of the logic of abduction may be seen as locally *glueing* breaks in the *continuum*, by means of an arsenal of methods which select effectively the "closer" explanatory hypotheses for a given break and which try to "erase" discontinuities from a new regularizing perspective:

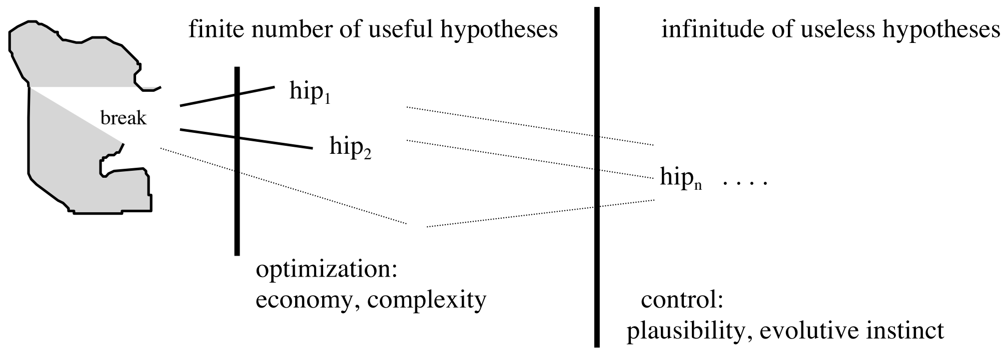


Figure 27.
Abduction as "glueing" breaks in the continuum.
"Optimal" selection of explanatory hypotheses.

Thus, the logic of abduction becomes in fact one of the basic supports of Peirce's pragmaticist architectonics and general synechism. Abduction serves as a regulatory system for the Real, for that plastic weaving (third) formed by facts (seconds) and hypotheses (firsts), where hypotheses are subject to complexity tests until they continuously fuse with facts. The logic of relatives –which, as we saw in our second chapter, filters technically continuity and generality– serves also as a crucial “filter” in the logic of abduction: it is the natural apparatus which provides the *normal forms*[193] of hypotheses, in order to study their adequate complexity.

Beyond Murray Murphey’s famous judgement[194] on the ineffective use of continuity to hold Peirce’s architectonics, we hope to have been able in this monograph on Peirce’s *continuum* to show that Peirce’s “castle” –very *real*, but not reducible to existence– is far from just flying in the air.

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[157] One of the basic *abductions* that supports our work on Peirce’s *continuum* contends that Peirce’s system constitutes a *natural* apparatus to *correlate* in a refined way global and local, total and partial, continuous and discrete. We think we have supported that hypothesis with enough *inductions* in our monograph. In this final paper, we are trying to support it through local bounded *deductions*.

[158] “Lowell Lectures” [1903; CP 4.512].

[159] "The Bedrock Beneath Pragmaticism” [1906; CP 4.561, note 1].

[160] “Prolegomena to an Apology for Pragmaticism” [c.1906; NE 4.324].

[161] “Logical Tracts” [c.1903; CP 4.448].

[162] We follow Peirce’s indication, brought up by Don Roberts: “*The Gamma Part* supposes the reasoner to invent for himself such additional kinds of signs as he may find desirable” (Ms. 693, cited in Don Roberts, *The Existential Graphs of Charles S. Peirce*, The Hague: Mouton, 1973, p.75). The thick identity line, representing second-order existential quantification, is such an “invention”.

[163] For full presentations of the existential graphs, one can consult Don Roberts, op.cit. (Ph.D. thesis, University of Illinois, 1963); Jay Zeman, *The Graphical Logic of C.S. Peirce*, Ph.D. Thesis, University of Chicago, 1964; Pierre Thibaud, *La logique de Charles Sanders Peirce: De l’algèbre aux graphes*, Aix-en-Provence: Université de Provence, 1975; or Robert Burch, *A Peircean Reduction Thesis. The Foundations of Topological Logic*, Lubbock: Texas Tech University Press, 1991.

[164] The proofs of equivalences ALPHA or BETA are far from being obvious: see Roberts or Zeman, op.cit. (conjectures due to Peirce, proofs to Roberts (1963) and Zeman (1964)). The best treatment of GAMMA modal systems is to be found in Zeman, op.cit., chapter III, “The Gamma Systems”, pp. 140-177. Zeman shows that the GAMMA calculus extending ALPHA to the broken cut *without restrictions* in the iteration and deiteration rules corresponds to a Lukasiewicz modal calculus, while other GAMMA extensions *with restrictions on iteration and deiteration through broken cuts* correspond to Lewis’ systems S4 and S5.

[165] “Come on, my Reader, and let us construct a diagram to illustrate the general course of thought”, in “Prolegomena to an Apology for Pragmaticism” [1906; CP 4.530].

[166] The passage “from essence to existence”, somewhat obscure in Heidegger’s philosophy, has been thoroughly studied in modern (1900-1940) mathematical creativity by Albert Lautman in his outstanding doctoral thesis “Essai sur les notions de structure et d’existence en mathématiques” (1937), in: Albert Lautman, *Essai sur l’unité des mathématiques et divers écrits*, Paris: Union Générale d’Éditions(10-18), 1977.

[167] “Nothing” in Veronese’s full intensional sense: a fluid primigenial *continuum*.

[168] [1903; PPM, *passim*]. See also: “Pragmatism” [1907; EP 2.398-433].

[169] The problem of the “proof of pragmaticism” has been one of the crucial open problems in peircean scholarship. See, for example, Richard S. Robin, “Classical pragmatism and pragmatism’s proof”, pp.145-146, in: Jacqueline Brunning, Paul Forster (eds.), *The Rule of Reason. The Philosophy of Charles Sanders Peirce*, Toronto: University of Toronto Press, 1997.

[170] “Prolegomena to an Apology for Pragmaticism” [1906; CP 4.534]. An immense majority of Peirce scholars considers “faulty” or mistaken the connections that Peirce sought between the existential graphs and proofs of pragmaticism (see, for example, Zeman, op.cit., p.177). Our position, instead,

seeks to retrieve and further advance the richness of those connections, following J. Esposito (*Evolutionary Metaphysics. The Development of Peirce's Theory of Categories*, Ohio: Ohio University Press, 1980, p. 228) who considered that the existential graphs "not only appear to establish the truth of the pragmatic maxim *philosophically* in the form of a deduction, but also *pragmatically* and *inductively* by affording an efficient logical system".

[171] "Phaneroscopy" [1906; CP 4.534, note 1].

[172] *Ibid*. [1906; CP 4.7]. The fact that existential graphs help to contemplate "with the wrong side out" the proof of pragmaticism can be interpreted as an indication that the proof has to be strongly *modalized* (as here we try). The *reverse* of the sheet of assertion is not just the world of non existence, but also the world of *possible* existence. In fact, the situation could further be enriched, if we could be able to implement Peirce's full geometry of the graphs: "Existential graphs (...) must be regarded only as *projection* upon (a) surface of a sign extended in three dimensions. *Three dimensions* are necessary and sufficient for the expression of all assertions" (Ms.654.6-7, cited in Esposito, op.cit., p. 227).

[173] In view of Dipert's sad and truthful comment, "It is a pity that logicians and philosophers have ceded so much of Peirce's work on "diagrams"" (R. Dipert, "Peirce's Underestimated Place in the History of Logic: A Response to Quine", in K. Ketner (ed.), *Peirce and Contemporary Thought: Philosophical Inquiries*, New York: Fordham University Press, 1995, p. 48), we try here to redress some logician's oversights long overdue.

[174] It should be observed that Peirce's system, and the progressive combinatorial bounds we propose, are very close to Leibniz's general project, clearly retrieved by XXth century mathematical logic.

[175] The "second step" (construction of diagrams) in the mathematical proof of a theorem is thus fulfilled. Peirce considered that the construction of diagrams could be "the weakest point in the whole demonstration" (Ms.1147.5^2). See D. Roberts, "An Introduction to Peirce's Proof of Pragmaticism", *Transactions of the Charles S. Peirce Society*, XIV (1978), p. 125. In our approach, after the diagram is constructed, experimentation, observation and deduction follow, as advocated by Peirce.

[176] In the horizontal order of illative inference, the specific uses of the rules are: erasure of lines of identity (beta, gamma) in an even area (6 nested cuts alpha *and* gamma); deiteration of identity lines beta and gamma to regions with lower number of cuts (4 around beta line, 1 around gamma line); erasure of the beta identity line in an even area (4 cuts) and apparition of the gamma second-order axiom (thick line unenclosed); deiteration of the all gamma identity line; erasure of the gamma line in an even area (0 cuts) and twice double alpha cut elimination.

[177] The problem lies in the first step of the deduction, which cannot be reversed: the *erasures* of the lines of identity in the 6 nested cuts area cannot be turned into *insertions* and glueings of extended new identity lines, for which we would need to be in an odd area.

[178] It should be observed that our methodology follows closely the pragmaticist maxim itself: to capture the *actual* maxim, it has been locally represented in a given context and therein his *necessary* logical status has been studied. Afterwards, we would have to think in all *possible* gamma systems of representation, in order to obtain a faithful reading of the maxim.

[179] For a different "continuity interpretation" of the graphs, see Jay Zeman, "Peirce's Graphs – the Continuity Interpretation", *Transactions of the Charles Sanders Peirce Society* 4 (1968), 144-154 (text corresponding to the introduction of Zeman's fundamental doctoral thesis, op.cit.).

[180] Roberts, op.cit., chapter 2 and appendix 1.

[181] "An Outline Classification of the Sciences" [1903; EP 2.260].

[182] Robin, op.cit., pp.145-146.

[183] Such a continuous unfolding of Peirce's classification of the sciences seems here to be hinted for the first time. In part, it corresponds to Pape's view that hypotheses should be considered as "singularities" in the space of continuous logical relations (H. Pape, "Abduction and the Topology of Human Cognition", *Transactions of the Charles S. Peirce Society* XXXV (1999), 248-269, particularly p. 250). We contend, in fact, that the discrete branching classification of the sciences may be seen as a sort of singularity, to be further embedded in the continuous space of gamma graphs. The embedding of the *discrete* triadic branching into *continuous* gamma graphs would also substantiate Hausman's forceful insight that *possibilia are loci of branching* (C. Hausman, *Charles Peirce's Evolutionary Philosophy*, New York: Cambridge University Press, 1993, pp. 185-189).

[184] "Letter to Paul Carus" - "Draft" [1909; manuscript in Max Fisch, *Peirce, Semeiotic and Pragmatism* (eds. Ketner, Kloesel), Bloomington: Indiana University Press, 1986, p.372].

[185] EP 2.398: "Since Peirce's conclusion amounts to a paraphrase of his definition of pragmatism, his proof [of pragmaticism] is complete".

[186] "Pragmatism" [1907; EP 2.418].

[187] "This step of adopting a hypothesis *as being suggested by the facts*, is what I call *abduction*", in: "On the Logic of Drawing History from Ancient Documents especially from Testimonies" [1901; HP 2.732] (first italics are ours).

[188] "Harvard Lectures" [1903; PPM 249].

[189] *Ibid*. [1903; PPM 245]. Abduction, as fully controlled logical inference, has been finally studied with all due rigour of contemporary mathematical logic in Atocha Aliseda-Llera, "Seeking Explanations: Abduction in Logic, Philosophy of Science and Artificial Intelligence", Ph.D. Thesis, Stanford University, 1997.

[190] "On the Logic of Drawing History from Ancient Documents especially from Testimonies" [1901; HP 2.753-754].

[191] It is the outstanding case of the *reverse mathematics* program (1970-2000) of Friedman and Simpson, which has located minimal and natural subsystems of second-order arithmetic fully *equivalent* (deduction and retroduction) to relatively complex theorems in mathematical practice (Bolzano-Weierstrass or Hahn-Banach, for example). See Stephen Simpson, *Subsystems of Second Order Arithmetic*, New York: Springer, 1999. Simpson's monograph was one of the more awaited texts in logic in the last two decades of the XXth century, and, once again, it harmonizes perfectly with many peircean motifs.

[192] "On the Logic of Drawing History from Ancient Documents especially from Testimonies" [1901; HP 2,726].

[193] Peirce's "cathedral" is eminently accumulative: the intuitions of the decade 1900-1910 on processes of abductive optimization rise over Peirce's deep work in the algebra of logic (1870-1885). "Normal forms" appear in the article "On the Algebra of Logic: A Contribution to the Philosophy of Notation" [1885; W 5.182-185], one of the most outstanding papers in all the history of logic. Peirce's thought is a *continuum* which fuses very diverse breaches in its evolution, *since 1859* ("Diagram of the IT" [1859; W 1.530], where a *diagram* draws, towards the future, the anatomy of its later modal triadization) *until 1911* ("Letter to A.D. Risteen" [1911; reference in Roberts, op.cit., p.135], where a *sketch* records, towards the past, the anatomy of its architectonics).

[194] "Peirce was never able to find a way to utilize the continuum concept effectively. The magnificent synthesis which the theory of continuity seemed to promise somehow always eluded him, and the shining vision of the great system always remained a castle in the air", in: Murray Murphey, *The Development of Peirce's Philosophy*, Cambridge: Harvard University Press, 1961, p.407. Consider, nevertheless, his new preface to the reissue of his pioneering work: "On some matters I was subsequently able to understand Peirce better, and this is particularly true of Peirce's later work. Peirce was more successful in achieving a coherent system than I thought in 1961" (2nd ed., Indianapolis: Hackett Publishing Co., 1993, p. v).